Path Following States of Proportional-Control Unicycles with Bearing-Only Sensing in Pursuit of a Constant Velocity Target

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November 18, 2021

Abstract

This report presents an analysis of the pursuit of a constant-velocity target by a unicycle agent moving at a constant speed, and guided by a steering control law proportional to the bearing angle towards the target. We categorize the system states and transitions between them to show that the pursuing agent must eventually either capture the target in finite time or asymptotically follow its path, regardless of initial conditions, given the gain in the control law is large enough.

1 Model Definition

Let a target's dynamics define a global frame such that the target's trajectory is described as

$$p_{target} = \begin{bmatrix} vt \\ 0 \end{bmatrix}, \tag{1}$$

t is time, and v is the target's speed. Consider a pursuing agent with Unicycle-Model kinematics

$$p_{agent} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{2}$$

where $p_{agent} = (x, y)^T$ is the pursuing agent's position, θ is the agent's orientation, and ω its turning rate, see Figure 1a. The relationship between the pursuing agent and the target can be described by the distance between them r, the bearing angle towards the target as measured from the agent's frame β, and the bearing angle towards the agent from the target's frame. In the settings proposed in this problem statement, the bearing angle towards the agent from the target's frame is equivalent to $\alpha - \pi$, where α is the azimuth from the agent to the target in the global frame, see Figure 1b.

Suppose we wanted the pursuing agent to capture or track the target by setting a proportional controller

$$\omega = \kappa\beta \tag{3}$$

where κ is a gain that amplifies β. We define "capture" as reaching a distance r_c from the target, and "tracking" as reaching a trajectory where, given arbitrary $\varepsilon > 0$, $\exists t_c \, | \forall t > t_c \, , \, |\beta(t)| < \varepsilon$ and $|\alpha(t)| < \varepsilon$.

Problem Statement: Given r_c and ε, find t_c such that either $\forall t > t_c$, $|\beta(t)| < \varepsilon$ and $|\alpha(t)| < \varepsilon$, or $\exists t < t_c |$ $r(t) \leq r_c$.

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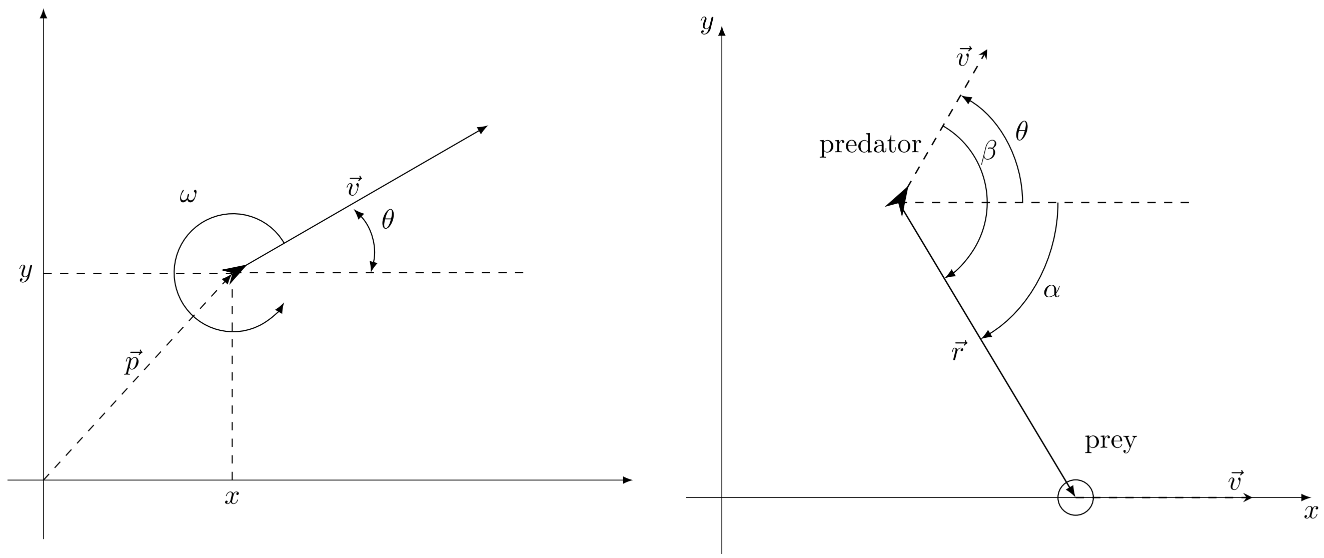


(a) The Unicycle Model

(b) The Agent-Target Relationship.

Figure 1: The Unicycle Pursuit Problem

2 Analysis

Let $p_0 = (x_0, y_0)^T$ be the pursuing agent's location at $t = 0$. Let α be the azimuth from the agent's location to the target's location,

$$\tan(\alpha) = \frac{y}{x - vt}; \tag{4}$$

and from Figure 1b,

$$\beta = \alpha - \theta. \tag{5}$$

Looking at the target's motion from a frame attached to the agent, with its real axis pointing in the direction of the target's velocity,

$$p_{target}^{agent} = re^{i\alpha}$$

where $r = \|\vec{r}\|$ is the distance form the target to the agent.

$$\frac{d}{dt} p_{target}^{agent} = (\dot{r} + ir\dot{\alpha})\, e^{i\alpha}.$$

The rotation component of the target's movement in the agent's perspective is therefore

$$r\dot{\alpha} = v\left(\sin(-\alpha) - \sin(\theta - \alpha)\right)$$

$$\Downarrow$$

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right), \tag{6}$$

and the rate of change in distance is

$$\dot{r} = v\left(\cos(-\alpha) - \cos(\theta - \alpha)\right);$$

$$\Downarrow$$

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right). \tag{7}$$

From Equations 3, 5, and 6,

$$\dot{\beta} = \dot{\alpha} - \dot{\theta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \omega$$

$$\Downarrow$$

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta \tag{8}$$

2.1 Prerequisites

Lemma 2.1. *If $\kappa > \frac{4}{3}\frac{v}{r_c}$, and $r(t_0) > r_c$, then $-\frac{\pi}{2} < \beta(t) < \frac{\pi}{2}\ \forall t \Big| t > t_0 + \frac{1}{\kappa}\ln\left(\frac{\beta(t_0) - 2\frac{v}{\kappa r_c}}{\frac{\pi}{2} - 2\frac{v}{\kappa r_c}}\right)$.*

Proof. From Equation 8,

$$-2\frac{v}{r_c} - \kappa\beta \le -2\frac{v}{r} - \kappa\beta \le \dot{\beta} \le 2\frac{v}{r} - \kappa\beta \le 2\frac{v}{r_c} - \kappa\beta$$

$$\Downarrow$$

$$\beta^-(t) \le \beta(t) \le \beta^+(t);$$

$$\beta^-(t) = -2\frac{v}{\kappa r_c} + \left(\beta(t_0) + 2\frac{v}{\kappa r_c}\right)e^{-\kappa(t-t_0)};$$

$$\beta^+(t) = 2\frac{v}{\kappa r_c} + \left(\beta(t_0) - 2\frac{v}{\kappa r_c}\right)e^{-\kappa(t-t_0)}.$$

If $\frac{\pi}{2} \le |\beta(t_0)| \le \pi$, then the magnitude of the bounds on β shrink asymptotically to $2\frac{v}{\kappa r_c} < \frac{\pi}{2}$, and reach $\frac{\pi}{2}$ by t_1,

$$\beta^+(t_1) = 2\frac{v}{\kappa r_c} + \left(\beta(t_0) - 2\frac{v}{\kappa r_c}\right)e^{-\kappa(t_1-t_0)} = \frac{\pi}{2}$$

$$\Downarrow$$

$$t_1 = t_0 + \frac{1}{\kappa}\ln\left(\frac{\beta(t_0) - 2\frac{v}{\kappa r_c}}{\frac{\pi}{2} - 2\frac{v}{\kappa r_c}}\right).$$

□

We designate the condition $\frac{\pi}{2} \le \beta \le \pi$ with a system state we call E, and the condition $-\pi < \beta \le -\frac{\pi}{2}$ with a system state we call E^-.

Lemma 2.2.

$$\alpha(t_0) = 0, r(t_0) > r_c \Rightarrow r(t) > r_c \forall t > t_0.$$

Proof. The target and pursuer agent have the same speed. If the agent is on the target's path ($\alpha = 0$), then the shortest path towards the target is on the target's straight path, therefore the best the agent can do in terms of pursuing the target is to stay on the straight path, resulting in a constant r and never capturing the target. Any other course of action taken by the agent will increase r, and again result in never capturing the target. □

2.2 Pursuit States

Lemma 2.3. *If $\kappa > \frac{v}{r_c}$, and at time $t = t_0$,*

1. $0 \le \alpha(t_0) < \beta(t_0) < \frac{\pi}{2}$,
2. $r(t_0) > r_c$;

then

1. $t_1 = t_0 + \frac{1}{\kappa}\ln\left(\frac{2}{1+\frac{\alpha(t_0)}{\beta(t_0)}}\right)$,
2. $\beta^+(t_1) = \beta(t_0)e^{-\left(\kappa - \frac{v}{r(t_0)}\right)(t_1-t_0)} - \frac{\frac{v}{r(t_0)}}{\kappa - \frac{v}{r(t_0)}}\sin(\alpha(t_0))\left(1 - e^{-\left(\kappa - \frac{v}{r(t_0)}\right)(t_1-t_0)}\right)$,
3. $\alpha^+(t_1) = \beta(t_0) + 2\,\mathrm{arccot}\left(\left(\cot\left(\frac{\alpha(t_0)-\beta(t_0)}{2}\right) - \tan(\beta(t_0))\right)e^{\frac{v}{r(t_0)}\cos(\beta(t_0))(t_1-t_0)} + \tan(\beta(t_0))\right)$,
4. $r^+(t_1) = r(t_0) + \frac{v}{\kappa}\ln\left(\frac{2}{1+\frac{\alpha(t_0)}{\beta(t_0)}}\right)(\cos(\alpha(t_0)) - \cos(\beta(t_0)))$,

5. $\alpha^{-}(t_1) = \alpha(t_0) + \frac{v}{\kappa r^{+}(t_1)} \ln\left(\frac{2}{1+\frac{\alpha(t_0)}{\beta(t_0)}}\right)\left(\sin\left(\frac{\beta(t_0)-\alpha(t_0)}{2}\right) - \sin\left(\alpha^{+}(t_1)\right)\right)$,

6. $t_2 = t_0 + \frac{r(t_0)}{\kappa r(t_0)-v} \ln\left(\frac{\beta(t_0)+\frac{\frac{v}{r(t_0)}}{\kappa-\frac{v}{r(t_0)}}\sin(\alpha(t_0))}{\alpha^{-}(t_1)+\frac{\frac{v}{r(t_0)}}{\kappa-\frac{v}{r(t_0)}}\sin(\alpha(t_0))}\right)$,

and $\exists t_* \,|t_1 \le t_* \le t_2$ *such that* $\alpha(t_*) = \beta(t_*) \le \beta^{+}(t_1)$.

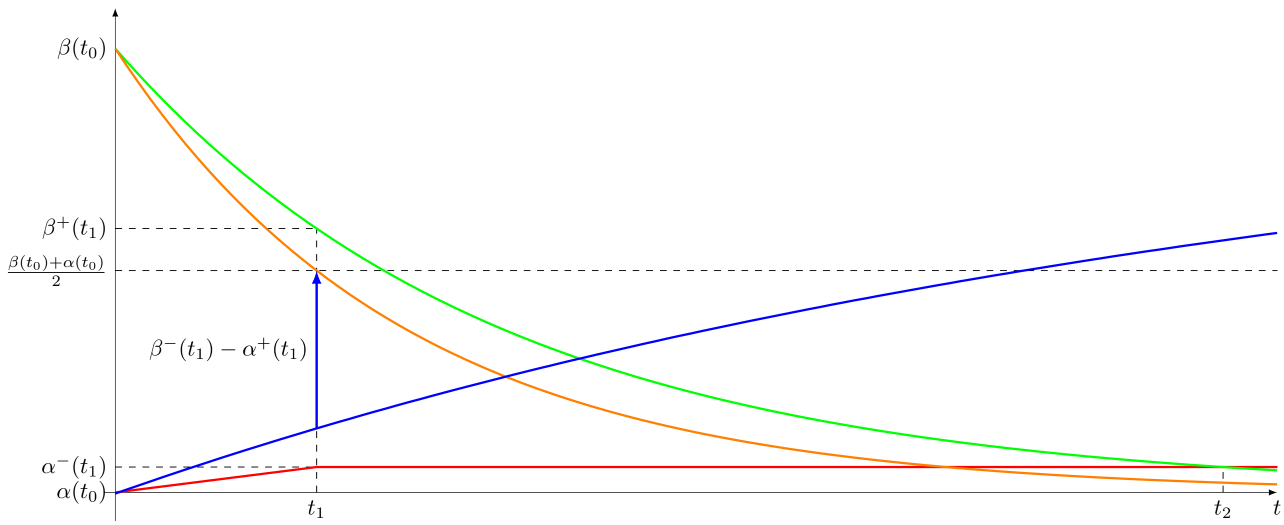


Figure 2: Lemma 2.3 proof outline. We start by finding β^+ and β^-, the bounds of β, and use β^- to calculate t_1 and $\beta^+(t_1)$. We then find α^+, an upper bound on α, and $\alpha^+(t_1)$. We continue to find the upper bound on $r(t_1)$, and with it we calculate $\alpha^-(t_1)$, the minimal possible value for α at time t_1. We conclude by finding time t_2, when $\beta^+(t_2) = \alpha^-(t_1)$, and note that $\beta = \alpha$ must happen between t_1 and t_2.

Figure 2 shows an outline of the proof. We designate the condition $0 \le \alpha(t) < \beta(t) < \frac{\pi}{2}$ with a system state we call A. Figure 7a shows α, β and their angular velocities while in state A.

Proof. While in state A, $\sin(\beta) > \sin(\alpha)$ and $\cos(\alpha) > \cos(\beta)$, therefore

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) > 0,$$

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) > 0,$$

and α and r grow accordingly.
Meanwhile, since $r(t_0) > r_c$, $\dot{r} > 0$, and $\kappa > \frac{v}{r_c} > \frac{v}{r(t_0)}$,

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta \le \frac{v}{r(t_0)}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta$$

$$< -\frac{v}{r(t_0)}\sin(\alpha) + \left(\frac{v}{r(t_0)} - \kappa\right)\beta < -\frac{v}{r(t_0)}\sin(\alpha(t_0)) - \left(\frac{\kappa r(t_0) - v}{r(t_0)}\right)\beta < 0.$$

So we see that while in state A, α grows and β shrinks.

$$-\kappa\beta < \dot{\beta} < -\frac{v}{r(t_0)}\sin(\alpha(t_0)) - \left(\kappa - \frac{v}{r(t_0)}\right)\beta$$

$$\Downarrow$$

$$\beta(t) > \beta^-(t) = \beta(t_0)e^{-\kappa(t-t_0)};$$

$$\beta(t) < \beta^+(t) = \beta(t_0)e^{-\left(\kappa-\frac{v}{r(t_0)}\right)(t-t_0)} - \frac{\frac{v}{r(t_0)}}{\kappa-\frac{v}{r(t_0)}}\sin(\alpha(t_0))\left(1 - e^{-\left(\kappa-\frac{v}{r(t_0)}\right)(t-t_0)}\right).$$

Since $\alpha(t_0) > 0$ and $\kappa > \frac{v}{r(t_0)}$, β^+ decays to a negative value.
Let t_1 such that

$$\beta^-(t_1) = \beta(t_0)e^{-\kappa(t_1-t_0)} = \frac{\beta(t_0)+\alpha(t_0)}{2}$$

$$\Downarrow$$

$$t_1 = t_0 + \frac{1}{\kappa}\ln\left(\frac{2}{1+\frac{\alpha(t_0)}{\beta(t_0)}}\right),$$

and $\beta^+(t_1) = \left(\beta(t_0) + \frac{\frac{v}{r(t_0)}}{\kappa-\frac{v}{r(t_0)}}\sin\left(\alpha(t_0)\right)\right)e^{-\left(\kappa-\frac{v}{r(t_0)}\right)(t_1-t_0)} - \frac{\frac{v}{r(t_0)}}{\kappa-\frac{v}{r(t_0)}}\sin\left(\alpha(t_0)\right)$.
We now find an upper bound on α,

$$\dot{\alpha} = \frac{v}{r}\left(\sin\left(\beta\right)-\sin\left(\alpha\right)\right) \leq \frac{v}{r(t_0)}\left(\sin\left(\beta(t_0)\right)-\sin\left(\alpha\right)\right);$$

$$\sin(\alpha) = 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) = \frac{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{\sin^2\left(\frac{\alpha}{2}\right)+\cos^2\left(\frac{\alpha}{2}\right)} = \frac{2\tan\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)};$$

$$\Downarrow$$

$$\dot{\alpha} \leq \frac{v}{r(t_0)}\sin\left(\beta(t_0)\right) - \frac{v}{r(t_0)}\frac{2\tan\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)}$$

$$\Downarrow$$

$$\alpha(t) \leq \alpha^+(t);$$

$$\dot{\alpha} \leq \dot{\alpha}^+ = \frac{v}{r(t_0)}\sin\left(\beta(t_0)\right) - \frac{v}{r(t_0)}\frac{2\tan\left(\frac{\alpha^+}{2}\right)}{1+\tan^2\left(\frac{\alpha^+}{2}\right)}.$$

Substituting $u = \tan\left(\frac{\alpha^+}{2}\right)$,

$$\frac{du}{d\alpha^+} = \frac{1+u^2}{2}$$

$$\Downarrow$$

$$d\alpha^+ = \frac{2du}{1+u^2}$$

$$\Downarrow$$

$$\frac{d\alpha^+}{dt} = \frac{2}{1+u^2}\frac{du}{dt}$$

$$\Downarrow$$

$$\frac{2}{1+u^2}\frac{du}{dt} = \frac{v}{r(t_0)}\left(\sin\left(\beta(t_0)\right) - \frac{2u}{1+u^2}\right)$$

$$\Downarrow$$

$$\frac{2}{1+u^2}\frac{du}{dt} = \frac{v\sin\left(\beta(t_0)\right)}{r(t_0)}\frac{u^2-\frac{2u}{\sin(\beta(t_0))}+1}{1+u^2} = \frac{v\sin\left(\beta(t_0)\right)}{r(t_0)}\frac{u^2-\frac{2u}{\sin(\beta(t_0))}+\frac{1}{\sin^2(\beta(t_0))}-\frac{1}{\sin^2(\beta(t_0))}+1}{1+u^2}$$

$$\Downarrow$$

$$\frac{2}{1+u^2}\frac{du}{dt} = \frac{v\sin\left(\beta(t_0)\right)}{r(t_0)}\frac{\left(u-\frac{1}{\sin(\beta(t_0))}\right)^2+1-\frac{1}{\sin^2(\beta(t_0))}}{1+u^2}$$

$$\Downarrow$$

$$\int\frac{du}{\left(u-\frac{1}{\sin(\beta(t_0))}\right)^2+1-\frac{1}{\sin^2(\beta(t_0))}} = \int\frac{v\sin\left(\beta(t_0)\right)}{2r(t_0)}dt$$

$$\Downarrow$$

$$\int \frac{du}{\left(u-\frac{1}{\sin(\beta(t_0))}\right)^2-\cot^2\left(\beta(t_0)\right)}=\int \frac{v\sin\left(\beta(t_0)\right)}{2r(t_0)}dt$$

$$\Downarrow$$

$$\frac{\tan\left(\beta(t_0)\right)}{2}\left(\int \frac{du}{\left(u-\frac{1}{\sin(\beta(t_0))}\right)-\cot\left(\beta(t_0)\right)}-\int \frac{du}{\left(u-\frac{1}{\sin(\beta(t_0))}\right)+\cot\left(\beta(t_0)\right)}\right)=\int \frac{v\sin\left(\beta(t_0)\right)}{2r(t_0)}dt$$

$$\Downarrow$$

$$\frac{\tan\left(\beta(t_0)\right)}{2}\left(\int \frac{du}{u-\frac{1+\cos(\beta(t_0))}{\sin(\beta(t_0))}}-\int \frac{du}{u-\frac{1-\cos(\beta(t_0))}{\sin(\beta(t_0))}}\right)=\int \frac{v\sin\left(\beta(t_0)\right)}{2r(t_0)}dt$$

$$\Downarrow$$

$$\int \frac{du}{u-\cot\left(\frac{\beta(t_0)}{2}\right)}-\int \frac{du}{u-\tan\left(\frac{\beta(t_0)}{2}\right)}=\int \frac{v}{r(t_0)}\cos\left(\beta(t_0)\right)dt$$

$$\Downarrow$$

$$\ln\left(u-\cot\left(\frac{\beta(t_0)}{2}\right)\right)-\ln\left(u-\tan\left(\frac{\beta(t_0)}{2}\right)\right)=\frac{v}{r(t_0)}\cos\left(\beta(t_0)\right)(t-t_0)+C_0$$

$$\Downarrow$$

$$\ln\left(\frac{u-\cot\left(\frac{\beta(t_0)}{2}\right)}{u-\tan\left(\frac{\beta(t_0)}{2}\right)}\right)=\frac{v}{r(t_0)}\cos\left(\beta(t_0)\right)(t-t_0)+C_0$$

$$\Downarrow$$

$$\ln\left(\frac{\tan\left(\frac{\alpha^+}{2}\right)-\cot\left(\frac{\beta(t_0)}{2}\right)}{\tan\left(\frac{\alpha^+}{2}\right)-\tan\left(\frac{\beta(t_0)}{2}\right)}\right)=\frac{v}{r(t_0)}\cos\left(\beta(t_0)\right)(t-t_0)+C_0$$

$$\Downarrow$$

$$\ln\left(\frac{\frac{\sin\left(\frac{\alpha^+}{2}\right)\sin\left(\frac{\beta(t_0)}{2}\right)-\cos\left(\frac{\alpha^+}{2}\right)\cos\left(\frac{\beta(t_0)}{2}\right)}{\cos\left(\frac{\alpha^+}{2}\right)\sin\left(\frac{\beta(t_0)}{2}\right)}}{\frac{\sin\left(\frac{\alpha^+}{2}\right)\cos\left(\frac{\beta(t_0)}{2}\right)-\sin\left(\frac{\beta(t_0)}{2}\right)\cos\left(\frac{\alpha^+}{2}\right)}{\cos\left(\frac{\alpha^+}{2}\right)\cos\left(\frac{\beta(t_0)}{2}\right)}}\right)=\frac{v}{r(t_0)}\cos\left(\beta(t_0)\right)(t-t_0)+C_0$$

$$\Downarrow$$

$$\ln\left(-\frac{\cos\left(\frac{\alpha^++\beta(t_0)}{2}\right)}{\sin\left(\frac{\alpha^+-\beta(t_0)}{2}\right)}\cot\left(\frac{\beta(t_0)}{2}\right)\right)=\frac{v}{r(t_0)}\cos\left(\beta(t_0)\right)(t-t_0)+C_0$$

$$\Downarrow$$

$$-\frac{\cos\left(\frac{\alpha^+-\beta(t_0)}{2}\right)\cos\left(\beta(t_0)\right)-\sin\left(\frac{\alpha^+-\beta(t_0)}{2}\right)\sin\left(\beta(t_0)\right)}{\sin\left(\frac{\alpha^+-\beta(t_0)}{2}\right)}\cot\left(\frac{\beta(t_0)}{2}\right)$$

$$=\left(\sin\left(\beta(t_0)\right)-\cot\left(\frac{\alpha^+-\beta(t_0)}{2}\right)\cos\left(\beta(t_0)\right)\right)\cot\left(\frac{\beta(t_0)}{2}\right)=C_1e^{\frac{v}{r(t_0)}\cos(\beta(t_0))(t-t_0)}$$

$$\Downarrow$$

$$\cot\left(\frac{\alpha^+-\beta(t_0)}{2}\right)=\tan\left(\beta(t_0)\right)+C_2e^{\frac{v}{r(t_0)}\cos(\beta(t_0))(t-t_0)}$$

$$\Downarrow$$

$$\alpha^+(t) = \beta(t_0) + 2\,\mathrm{arccot}\left(C_2 e^{\frac{v}{r(t_0)}\cos(\beta(t_0))(t-t_0)} + \tan\left(\beta(t_0)\right)\right);$$

$$\alpha^+(t_0) = \alpha(t_0)$$

$$\Downarrow$$

$$C_2 = \cot\left(\frac{\alpha(t_0) - \beta(t_0)}{2}\right) - \tan\left(\beta(t_0)\right)$$

$$\Downarrow$$

$$\alpha(t) \le \alpha^+(t),$$

$$\alpha^+(t) = \beta(t_0) + 2\,\mathrm{arccot}\left(\left(\cot\left(\frac{\alpha(t_0) - \beta(t_0)}{2}\right) - \tan\left(\beta(t_0)\right)\right) e^{\frac{v}{r(t_0)}\cos(\beta(t_0))(t-t_0)} + \tan\left(\beta(t_0)\right)\right).$$

For $t_0 \le t \le t_1$,

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) \le v\left(\cos(\alpha(t_0)) - \cos(\beta(t_0))\right)$$

$$\Downarrow$$

$$r(t_1) \le r(t_0) + v\left(\cos(\alpha(t_0)) - \cos(\beta(t_0))\right)(t_1 - t_0)$$

$$r(t_0) + \frac{v}{\kappa}\ln\left(\frac{2}{1 + \frac{\alpha(t_0)}{\beta(t_0)}}\right)\left(\cos(\alpha(t_0)) - \cos(\beta(t_0))\right) = r^+(t_1).$$

From these results we can find a lower bound on $\alpha(t)$, for all $t_0 < t < t_1$,

$$\dot{\alpha} \ge \frac{v}{r^+(t_1)}\left(\sin\left(\beta^-(t_1)\right) - \sin\left(\alpha^+(t_1)\right)\right)$$

$$\Downarrow$$

$$\alpha(t_1) \ge \alpha^-(t_1) = \alpha(t_0) + \frac{v}{r^+(t_1)}\left(\sin\left(\beta^-(t_1)\right) - \sin\left(\alpha^+(t_1)\right)\right)(t_1 - t_0)$$

Let t_2 be the moment at which $\beta^+(t_2) = \alpha^-(t_1)$,

$$\alpha^-(t_1) = -\frac{\frac{v}{r(t_0)}}{\kappa - \frac{v}{r(t_0)}}\sin\left(\alpha(t_0)\right) + \left(\beta(t_0) + \frac{\frac{v}{r(t_0)}}{\kappa - \frac{v}{r(t_0)}}\sin\left(\alpha(t_0)\right)\right) e^{-\left(\kappa - \frac{v}{r(t_0)}\right)(t_2 - t_0)}$$

$$t_2 = t_0 + \frac{1}{\kappa - \frac{v}{r(t_0)}}\ln\left(\frac{\beta(t_0) + \frac{\frac{v}{r(t_0)}}{\kappa - \frac{v}{r(t_0)}}\sin\left(\alpha(t_0)\right)}{\alpha^-(t_1) + \frac{\frac{v}{r(t_0)}}{\kappa - \frac{v}{r(t_0)}}\sin\left(\alpha(t_0)\right)}\right)$$

and $\exists t_*$, $t_1 \le t_* \le t_2$, such that $\beta(t_*) = \alpha(t_*)$. □

Corollary 2.4. *If $\kappa > \frac{v}{r_c}$, and at time $t = t_0$,*

1. $0 \le \alpha(t_0) < \beta(t_0) < \frac{\pi}{2}$,
2. $r(t_0) > r_c$;

then

1. $t_1 = t_0 + \frac{1}{\kappa}\ln\left(\frac{2}{1 + \frac{\alpha(t_0)}{\beta(t_0)}}\right)$,
2. $\beta^+(t_1) = \beta(t_0) e^{-\left(\kappa - \frac{v}{r(t_0)}\right)(t_1 - t_0)} - \frac{\frac{v}{r(t_0)}}{\kappa - \frac{v}{r(t_0)}}\sin\left(\alpha(t_0)\right)\left(1 - e^{-\left(\kappa - \frac{v}{r(t_0)}\right)(t_1 - t_0)}\right)$,
3. $\alpha^+(t_1) = \beta(t_0) + 2\,\mathrm{arccot}\left(\left(\cot\left(\frac{\alpha(t_0) - \beta(t_0)}{2}\right) - \tan\left(\beta(t_0)\right)\right) e^{\frac{v}{r(t_0)}\cos(\beta(t_0))(t_1 - t_0)} + \tan\left(\beta(t_0)\right)\right)$,

4. $r^{+}(t_1) = r(t_0) + \frac{v}{\kappa}\ln\left(\frac{2}{1+\frac{\alpha(t_0)}{\beta(t_0)}}\right)(\cos(\alpha(t_0)) - \cos(\beta(t_0)))$,

5. $\alpha^{-}(t_1) = \alpha(t_0) + \frac{v}{\kappa r^{+}(t_1)}\ln\left(\frac{2}{1+\frac{\alpha(t_0)}{\beta(t_0)}}\right)\left(\sin\left(\frac{\beta(t_0)-\alpha(t_0)}{2}\right) - \sin\left(\alpha^{+}(t_1)\right)\right)$,

6. $t_2 = t_0 + \frac{r(t_0)}{\kappa r(t_0)-v}\ln\left(\frac{\beta(t_0)+\frac{\frac{v}{r(t_0)}}{\kappa-\frac{v}{r(t_0)}}\sin(\alpha(t_0))}{\alpha^{-}(t_1)+\frac{\frac{v}{r(t_0)}}{\kappa-\frac{v}{r(t_0)}}\sin(\alpha(t_0))}\right)$,

and $\exists t_* \,|t_1 \leq t_* \leq t_2$ *such that* $\alpha(t_*) = \beta(t_*) \geq \beta^{+}(t_1)$.

We designate the condition $-\frac{\pi}{2} < \beta(t) < \alpha(t) \leq 0$ with a system state we call A^{-}.

Lemma 2.5. *If at time* $t = t_0$

1. $0 < \beta(t_0) \leq \alpha(t_0) < \frac{\pi}{2}$,
2. $r(t_0) > r_c$;

then

1. $r^{-}(t_0 + \frac{1}{\kappa}) = r(t_0) - \frac{v}{\kappa}(1 - \cos(\alpha(t_0)))$,
2. $a = \kappa\frac{v}{r^{-}(t_0+\frac{1}{\kappa})} = \frac{v}{r(t_0)}\frac{1}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}$,
3. $b = \frac{\beta(t_0)+a\left(\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\frac{\beta(t_0)}{e}+a\left(\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)\right)}$,
4. $t_2 = t_0 + \frac{1}{\kappa}\ln(b)$,
5. $\alpha^{-}(t_2) = \alpha(t_0) + \left(\frac{1}{e} + b\right)\beta(t_0) + a\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)(1 - b - \ln(b))$,
6. $\beta^{+}(t_2) = \frac{\beta(t_0)}{b}$,
7. $t_3 = t_2 - \frac{1}{\kappa}\ln\left(\frac{\frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^{+}(t_2)}{2}\right)\sin\left(\frac{\alpha^{-}(t_2)-\beta^{+}(t_2)}{2}\right)}{\beta^{+}(t_2)+\frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^{+}(t_2)}{2}\right)\sin\left(\frac{\alpha^{-}(t_2)-\beta^{+}(t_2)}{2}\right)}\right)$;

and either $\exists t_* \,|t_2 \leq t_* \leq t_3$ *such that* $\beta(t_*) = 0$, *or* $\exists t_* \,|t_0 < t_* \leq t_3$ *such that* $r(t_*) \leq r_c$.

Figure 3 shows an outline of the proof. We designate the condition $0 < \beta(t) \leq \alpha(t) < \frac{\pi}{2}$ with a system state we call B. Figure 7b shows α, β and their angular velocities while in state B.

Proof. While in state B, $0 < \beta(t) \leq \alpha(t) \leq \frac{\pi}{2}$, and

$$\dot{\alpha} = \frac{v}{r}(\sin(\beta) - \sin(\alpha)) \leq 0,$$

$$\dot{\beta} = \dot{\alpha} - \kappa\beta < \dot{\alpha} \leq 0,$$

therefore β shrinks faster than α. Also,

$$\dot{r} = v(\cos(\alpha) - \cos(\beta)) \leq 0,$$

shrinking r.

We will now find t_1, t_2, the upper and lower bounds respectively to the time required until $\beta = \frac{\beta(t_0)}{e}$.

$$\dot{\beta} \leq -\kappa\beta$$

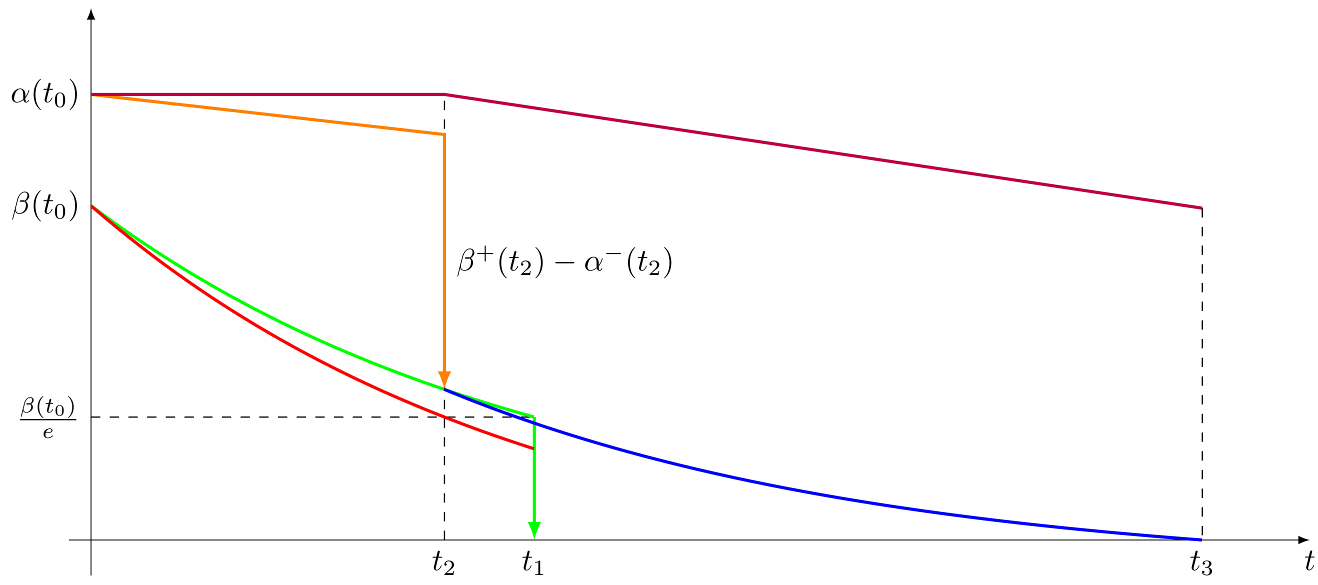


Figure 3: Lemma 2.5 proof outline. We start by finding an upper bound for β and using it to calculate t_1. We then use t_1 to find $r^-(t_1)$, which is needed in order to find the lower bound on β. We then proceed to calculate t_2, and find the lower bound on $\alpha(t_2)$. We conclude the proof by showing that the upper bound on β must reach 0 by t_3, and that therefore $\beta = 0$ must have happened between t_2 and t_3.

$$\Downarrow$$

$$\beta(t) \leq \beta(t_0)e^{-\kappa(t-t_0)} = \beta^+(t).$$

Let t_1 be the moment at which the upper bound on β reaches $\frac{\beta(t_0)}{e}$,

$$\beta^+(t_1) = \beta(t_0)e^{-\kappa(t_1-t_0)} = \frac{\beta(t_0)}{e}$$

$$\Downarrow$$

$$t_1 = t_0 + \frac{1}{\kappa}.$$

So $\frac{1}{\kappa}$ is the longest possible period of time required for $\beta \leq \frac{\beta(t_0)}{e}$.
While in state B,

$$\dot{r} = v\left(\cos\left(\alpha\right) - \cos(\beta)\right) > v\left(\cos\left(\alpha(t_0)\right) - 1\right),$$

and the lower bound on r is

$$r^-(t) = r(t_0) + v\left(\cos\left(\alpha(t_0)\right) - 1\right)(t - t_0) < r(t).$$

Using $r^-(t) < r(t)\ \forall t\,|t_0 < t \leq t_1$, the lower bound to the minimal possible value for r at t_1 becomes

$$r^-(t_1) = r(t_0) + v\left(\cos\left(\alpha(t_0)\right) - 1\right)(t_1 - t_0) = r(t_0) - \frac{v}{\kappa}\left(1 - \cos\left(\alpha(t_0)\right)\right).$$

We can now calculate the lower bound on $\beta(t)\ \forall t\,|t_0 < t < t_1$,

$$\dot{\beta} > \frac{v}{r^-(t_1)}\left(\sin\left(\frac{\beta(t_0)}{e}\right) - \sin\left(\alpha(t_0)\right)\right) - \kappa\beta$$

$$\Downarrow$$

$$\beta > \beta^-(t) = -\frac{v}{r(t_0)}\frac{\sin\left(\alpha(t_0)\right) - \sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa - \frac{v}{r(t_0)}\left(1 - \cos\left(\alpha(t_0)\right)\right)} + \left(\beta(t_0) + \frac{v}{r(t_0)}\frac{\sin\left(\alpha(t_0)\right) - \sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa - \frac{v}{r(t_0)}\left(1 - \cos\left(\alpha(t_0)\right)\right)}\right)e^{-\kappa(t-t_0)}$$

and find t_2, the first possible moment at which $\beta = \frac{\beta(t_0)}{e}$.

$$\beta^-(t_2) = -\frac{\frac{v}{r(t_0)}\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}$$

$$+\left(\beta(t_0) + \frac{\frac{v}{r(t_0)}\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}\right)e^{-\kappa(t_2-t_0)} = \frac{\beta(t_0)}{e}$$

$$\Downarrow$$

$$t_2 = t_0 + \frac{1}{\kappa}\ln\left(\frac{\beta(t_0) + \frac{v}{r(t_0)}\frac{\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}}{\frac{\beta(t_0)}{e} + \frac{v}{r(t_0)}\frac{\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}}\right).$$

During this time, $t_0 < t < t_2$, β shrunk faster than α and the two angles drifted away from one another.

$$\frac{d}{dt}(\alpha - \beta) = \dot{\alpha} - \dot{\beta} = \kappa\beta;$$

$$\kappa\beta^-(t) < \kappa\beta \le \kappa\beta^+(t) = \kappa\beta(t_0)e^{-\kappa(t-t_0)}$$

$$\Downarrow$$

$$-\kappa\frac{\frac{v}{r(t_0)}\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}$$

$$+\left(\beta(t_0) + \frac{\frac{v}{r(t_0)}\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}\right)\kappa e^{-\kappa(t-t_0)} < \frac{d}{dt}\left(\dot{\alpha} - \dot{\beta}\right)$$

$$\Downarrow$$

$$\alpha(t) - \beta(t) > \alpha(t_0) + \frac{\frac{v}{r(t_0)}\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}\left(1 - \kappa(t-t_0) - e^{-\kappa(t-t_0)}\right) - \beta(t_0)e^{-\kappa(t-t_0)}$$

and we can now calculate the lower bound on $\alpha(t_2) - \beta(t_2)$,

$$\alpha(t_2) - \beta(t_2) > \alpha(t_0) + \frac{\frac{v}{r(t_0)}\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}\left(1 - \kappa(t-t_0) - e^{-\kappa(t-t_0)}\right) - \beta(t_0)e^{-\kappa(t-t_0)}$$

$$\Downarrow$$

$$\alpha(t_2) > \alpha(t_0) + \beta(t_2) + \frac{\frac{v}{r(t_0)}\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}\left(1 - \kappa(t_2-t_0) - e^{-\kappa(t_2-t_0)}\right) - \beta(t_0)e^{-\kappa(t_2-t_0)}$$

$$= \alpha(t_0) + \beta(t_2) + \frac{\frac{v}{r(t_0)}\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}\left(1 + \ln\left(\frac{\frac{\beta(t_0)}{e} + \frac{v}{r(t_0)}\frac{\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}}{\beta(t_0) + \frac{v}{r(t_0)}\frac{\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}}\right)\right)$$

$$-\left(\frac{\frac{v}{r(t_0)}\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))} + \beta(t_0)\right)\left(\frac{\frac{\beta(t_0)}{e} + \frac{v}{r(t_0)}\frac{\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}}{\beta(t_0) + \frac{v}{r(t_0)}\frac{\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}}\right)$$

and

$$\alpha(t_2) \geq \alpha(t_0) + \frac{\beta(t_0)}{e} + \frac{\frac{v}{r(t_0)}\left(\sin\left(\alpha(t_0)\right) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}\left(1-\cos\left(\alpha(t_0)\right)\right)}\left(1 + \ln\left(\frac{\frac{\beta(t_0)}{e} + \frac{v}{r(t_0)}\frac{\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}}{\beta(t_0) + \frac{v}{r(t_0)}\frac{\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}}\right)\right)$$

$$-\left(\frac{\frac{v}{r(t_0)}\left(\sin\left(\alpha(t_0)\right) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\kappa - \frac{v}{r(t_0)}\left(1-\cos\left(\alpha(t_0)\right)\right)} + \beta(t_0)\right)\left(\frac{\frac{\beta(t_0)}{e} + \frac{v}{r(t_0)}\frac{\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}}{\beta(t_0) + \frac{v}{r(t_0)}\frac{\sin(\alpha(t_0))-\sin\left(\frac{\beta(t_0)}{e}\right)}{\kappa-\frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}}\right) = \alpha^-(t_2)$$

After t_2, and since $\alpha - \beta$ grows while α and β independently shrink in state B, we can calculate new upper bounds.

$$\dot{\alpha} = -\frac{v}{r}\left(\sin\left(\alpha\right) - \sin\left(\beta\right)\right) = -2\frac{v}{r}\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$< -2\frac{v}{r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^+(t_2)}{2}\right)\sin\left(\frac{\alpha^-(t_2)-\beta^+(t_2)}{2}\right) < 0;$$

$$\dot{\beta} = \dot{\alpha} - \kappa\beta < -2\frac{v}{r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^+(t_2)}{2}\right)\sin\left(\frac{\alpha^-(t_2)-\beta^+(t_2)}{2}\right) - \kappa\beta < 0,$$

and $\forall t \geq t_2$,

$$\alpha(t) < \alpha^+(t) = \alpha(t_0) - \frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^+(t_2)}{2}\right)\sin\left(\frac{\alpha^-(t_2)-\beta^+(t_2)}{2}\right)(t - t_2);$$

$$\beta(t) < \beta_2^+(t) = -\frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^+(t_2)}{2}\right)\sin\left(\frac{\alpha^-(t_2)-\beta^+(t_2)}{2}\right)$$
$$+\left(\beta^+(t_2) + \frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^+(t_2)}{2}\right)\sin\left(\frac{\alpha^-(t_2)-\beta^+(t_2)}{2}\right)\right)e^{-\kappa(t-t_2)},$$

and now we can find the time t_3, when $\beta_2^+(t_3) = 0$.

$$\beta_2^+(t_3) = -\frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^+(t_2)}{2}\right)\sin\left(\frac{\alpha^-(t_2)-\beta^+(t_2)}{2}\right)$$

$$+\left(\beta^+(t_2) + \frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^+(t_2)}{2}\right)\sin\left(\frac{\alpha^-(t_2)-\beta^+(t_2)}{2}\right)\right)e^{-\kappa(t_3-t_2)} = 0$$

$$\Downarrow$$

$$t_3 = t_2 - \frac{1}{\kappa}\ln\left(\frac{\frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^+(t_2)}{2}\right)\sin\left(\frac{\alpha^-(t_2)-\beta^+(t_2)}{2}\right)}{\beta^+(t_2) + \frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^+(t_2)}{2}\right)\sin\left(\frac{\alpha^-(t_2)-\beta^+(t_2)}{2}\right)}\right)$$

To conclude, since $0 < \beta < \beta_2^+$ while in state B and past t_2, and since $\beta_2^+(t_3) = 0$, then $\exists t\,|t_2 \leq t \leq t_3$ such that $\beta(t) = 0$. By t_3, the lowest possible value for $r(t)$ is

$$r^-(t_3) = r(t_0) - v\left(1 - \cos(\alpha(t_0))\right)(t_3 - t_0).$$

□

Corollary 2.6. *If at time* $t = t_0$

1. $-\frac{\pi}{2} < \alpha(t_0) \leq \beta(t_0) < 0,$
2. $r(t_0) > r_c;$

then

1. $r^{-}(t_0 + \frac{1}{\kappa}) = r(t_0) - \frac{v}{\kappa}\left(1 - \cos\left(\alpha(t_0)\right)\right)$,

2. $a = \kappa \frac{v}{r^{-}(t_0+\frac{1}{\kappa})} = \frac{v}{r(t_0)} \frac{1}{\kappa - \frac{v}{r(t_0)}(1-\cos(\alpha(t_0)))}$,

3. $b = \frac{\beta(t_0) + a\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}{\frac{\beta(t_0)}{e} + a\left(\sin(\alpha(t_0)) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)}$,

4. $t_2 = t_0 + \frac{1}{\kappa}\ln(b)$,

5. $\alpha^{-}(t_2) = \alpha(t_0) + \left(\frac{1}{e} + b\right)\beta(t_0) + a\left(\sin\left(\alpha(t_0)\right) - \sin\left(\frac{\beta(t_0)}{e}\right)\right)\left(1 - b - \ln(b)\right)$,

6. $\beta^{+}(t_2) = \frac{\beta(t_0)}{b}$,

7. $t_3 = t_2 - \frac{1}{\kappa}\ln\left(\frac{\frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^{+}(t_2)}{2}\right)\sin\left(\frac{\alpha^{-}(t_2)-\beta^{+}(t_2)}{2}\right)}{\beta^{+}(t_2) + \frac{2v}{\kappa r(t_0)}\cos\left(\frac{\alpha(t_0)+\beta^{+}(t_2)}{2}\right)\sin\left(\frac{\alpha^{-}(t_2)-\beta^{+}(t_2)}{2}\right)}\right)$;

and either $\exists t_* \,| t_2 \le t_* \le t_3$ *such that* $\beta(t_*) = 0$, *or* $\exists t_* \,| t_0 < t_* \le t_3$ *such that* $r(t_*) \le r_c$.

We designate the condition $-\frac{\pi}{2} < \alpha(t) \le \beta(t) < 0$ with a system state we call B^{-}.

Lemma 2.7. *If* $\kappa > 2\frac{v}{r_c}$, *and at time* $t = t_0$, $0 \le -\beta(t_0) < \alpha(t_0) < \frac{\pi}{2}$, *then*

1. $t_1 = t_0 + \frac{r(t_0)}{v}\ln\left(\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{v}{\kappa r_c}\frac{\alpha(t_0)-\beta(t_0)}{2}\right)}\right)$,

2. $t_2 = t_0 + \frac{r_c}{2v}\ln\left(\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{v}{\kappa r_c}\frac{\alpha(t_0)-\beta(t_0)}{2}\right)}\right)$,

3. $t3 = t_0 + \frac{r_c}{2v}\ln\left(-\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{\beta(t_0)}{2}\right)}\right)$;

and either

1. $0 \le -\beta(t_1) < \alpha(t_1) < \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right)$, *or*

2. $\exists t_*,\ t_2 < t_* \le t_1 \,|\, -\beta(t_*) = \alpha(t_*)$, *and if* $\beta(t_0) \le \frac{v}{r_c}\frac{\alpha(t_0)}{\frac{v}{r_c}-\kappa}$, *then* $0 < \alpha(t_*) < \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right)$ *or*

3. $\exists t_*,\ t_3 < t_* \le t_1 \,|\, -\beta(t_*) = \alpha(t_*)$, *and if* $\beta(t_0) > \frac{v}{r_c}\frac{\alpha(t_0)}{\frac{v}{r_c}-\kappa}$, *then*

$$0 < \alpha(t_*) < \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right) - \left(\beta(t_0) + \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right)\right) e^{-\frac{\kappa}{2}\frac{r_c}{v}\ln\left(-\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{\beta(t_0)}{2}\right)}\right)}$$

   or

4. $\exists t_*,\ t_0 \le t_* \le t_1 \,|\, r(t_*) \le r_c$.

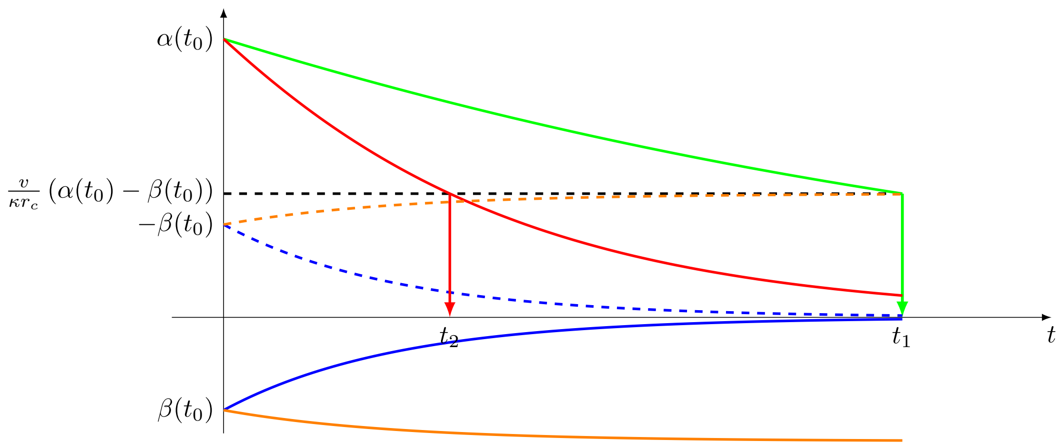
We designate the condition $0 \le -\beta < \alpha < \frac{\pi}{2}$ with a system state we call C. Figure 7c shows α, β and their angular velocities while in state C. Figure 4 shows an outline of the proof.

Proof. While in state C, $0 \le -\beta < \alpha \le \frac{\pi}{2}$, and

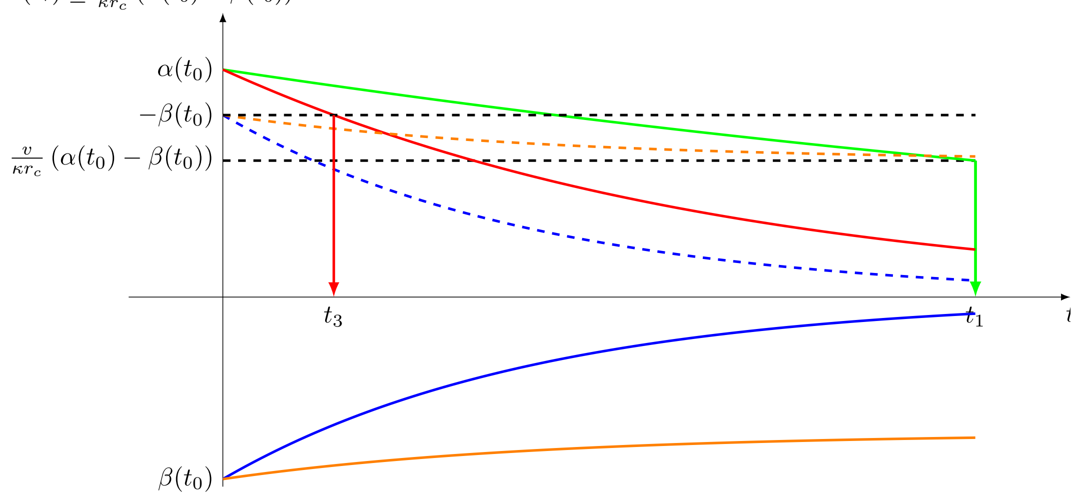
$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) < 0,$$

$$\dot{\alpha} - \dot{\beta} = \kappa\beta \le 0,$$

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) \le -\frac{v}{r}\sin(\alpha) < -\frac{v}{r(t_0)}\sin(\alpha) < 0,$$



(a) If $\beta(t_0) \leq \frac{v}{\kappa r_c}(\beta(t_0)-\alpha(t_0))$ and exiting the state with $\alpha(t_*) = -\beta(t_*)$, then $t_2 \leq t_* \leq t_1$ and $\alpha(t_*) \leq \frac{v}{\kappa r_c}(\alpha(t_0)-\beta(t_0))$



(b) If $\beta(t_0) > \frac{v}{\kappa r_c}(\beta(t_0)-\alpha(t_0))$ and exiting the state with $\alpha(t_*) = -\beta(t_*)$, then $t_3 \leq t_* \leq t_1$ and $\alpha(t_*) < -\beta^-(t_3)$

Figure 4: Lemma 2.7 proof outline. We find lower and upper bounds on α and β, and use the bounds on α to calculate t_1, t_2, and t_3. By t_1, either the system remains in state C or exits the state with $\alpha = -\beta$, or $r = r_c$.

In other words, r, α and $\alpha - \beta$ are positive and shrinking.

$$\sin(\alpha) = 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) = \frac{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{\sin^2\left(\frac{\alpha}{2}\right)+\cos^2\left(\frac{\alpha}{2}\right)} = \frac{2\tan\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)}.$$

$$\Downarrow$$

$$\dot{\alpha} < -\frac{v}{r(t_0)}\frac{2\tan\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)}.$$

Substituting $u = \tan\left(\frac{\alpha}{2}\right)$,

$$\frac{du}{d\alpha} = \frac{1+u^2}{2}$$

$$\Downarrow$$

$$d\alpha = \frac{2du}{1+u^2}$$

$$\Downarrow$$

$$\frac{d\alpha}{dt} = \frac{2}{1+u^2}\frac{du}{dt}$$

$$\Downarrow$$

$$\frac{2}{1+u^2}\frac{du}{dt} < -\frac{v}{r(t_0)}\frac{2u}{1+u^2}$$

$$\Downarrow$$

$$\int \frac{du}{u} < -\frac{v}{r(t_0)}\int dt$$

$$\Downarrow$$

$$\ln(u) < C_1 - \frac{v}{r_1(t_0)}(t-t_0)$$

$$\Downarrow$$

$$\tan\left(\frac{\alpha}{2}\right) < \tan\left(\frac{\alpha(t_0)}{2}\right)e^{-\frac{v}{r(t_0)}(t-t_0)}$$

$$\Downarrow$$

$$\alpha(t) < \alpha^+(t) = 2\arctan\left(\tan\left(\frac{\alpha(t_0)}{2}\right)e^{-\frac{v}{r(t_0)}(t-t_0)}\right).$$

Now that we found the upper bound on α, we can find the last possible moment in time when

$$\alpha(t) = \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right).$$

$$\alpha^+(t_1) = \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right)$$

$$\Downarrow$$

$$2\arctan\left(\tan\left(\frac{\alpha(t_0)}{2}\right)e^{-\frac{v}{r(t_0)}(t_1-t_0)}\right) = \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right)$$

$$\Downarrow$$

$$t_1 = t_0 + \frac{r(t_0)}{v}\ln\left(\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{v}{\kappa r_c}\frac{\alpha(t_0)-\beta(t_0)}{2}\right)}\right).$$

We continue and calculate a lower bound on α,

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) \geq -2\frac{v}{r_c}\sin(\alpha)$$

$$\Downarrow$$

$$\alpha(t) \geq \alpha^-(t) = 2\arctan\left(\tan\left(\frac{\alpha(t_0)}{2}\right)e^{-\frac{2v}{r_c}(t-t_0)}\right),$$

and the bounds on β,

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta \leq -\kappa\beta$$

$$\Downarrow$$

$$\beta(t) \leq \beta^+(t) = \beta(t_0)e^{-\kappa(t-t_0)}$$

and

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta = -2\frac{v}{r}\left(\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right) - \kappa\beta$$

$$\Downarrow$$

$$\dot{\beta} \geq -2\frac{v}{r_c}\sin\left(\frac{\alpha(t_0)-\beta(t_0)}{2}\right) - \kappa\beta \geq -2\frac{v}{r_c}\left(\frac{\alpha(t_0)-\beta(t_0)}{2}\right) - \kappa\beta$$

$$\Downarrow$$

$$\beta(t) \geq \beta^{-}(t) = -\frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right) + \left(\beta(t_0) + \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right)\right) e^{-\kappa(t-t_0)}.$$

Next, we calculate time t_2, when

$$\alpha^{-}(t_2) = \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right)$$

$$\Downarrow$$

$$2\arctan\left(\tan\left(\frac{\alpha(t_0)}{2}\right) e^{-\frac{2v}{r_c}(t2-t_0)}\right) = \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right)$$

$$\Downarrow$$

$$t2 = t_0 + \frac{r_c}{2v}\ln\left(\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{v}{\kappa r_c}\frac{\alpha(t_0)-\beta(t_0)}{2}\right)}\right),$$

and t_3, when

$$\alpha^{-}(t_3) = -\beta(t_0)$$

$$\Downarrow$$

$$2\arctan\left(\tan\left(\frac{\alpha(t_0)}{2}\right) e^{-\frac{2v}{r_c}(t3-t_0)}\right) = -\beta(t_0)$$

$$\Downarrow$$

$$t3 = t_0 + \frac{r_c}{2v}\ln\left(-\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{\beta(t_0)}{2}\right)}\right).$$

If $\beta(t_0) \leq \frac{v}{r_c}\frac{\alpha(t_0)}{\frac{v}{r_c}-\kappa}$, then $\forall t_0 \leq t < t_2$ it is guaranteed that $-\beta(t) \leq \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right) < \alpha(t)$, and therefore if $\exists t_* \mid \alpha(t_*) = \beta(t_*)$, then $\alpha(t_*) \leq \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right)$. Similarly, if $\beta(t_0) > \frac{v}{r_c}\frac{\alpha(t_0)}{\frac{v}{r_c}-\kappa}$, then $\forall t$, $-\dot{\beta}^{-}(t) < 0$ and therefore if $\exists t_* \mid \alpha(t_*) = \beta(t_*)$, then $\alpha(t_*) \leq -\beta^{-}(t_3)$;

$$\beta^{-}(t_3) = -\frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right) + \left(\beta(t_0) + \frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right)\right) e^{-\frac{\kappa}{2}\frac{r_c}{v}\ln\left(-\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{\beta(t_0)}{2}\right)}\right)}.$$

□

Corollary 2.8. *If $\kappa > 2\frac{v}{r_c}$, and at time $t = t_0$, $0 \leq \beta(t_0) < -\alpha(t_0) < \frac{\pi}{2}$, then*

1. $t_1 = t_0 + \frac{r(t_0)}{v}\ln\left(\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{v}{\kappa r_c}\frac{\alpha(t_0)-\beta(t_0)}{2}\right)}\right)$,
2. $t_2 = t_0 + \frac{r_c}{2v}\ln\left(\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{v}{\kappa r_c}\frac{\alpha(t_0)-\beta(t_0)}{2}\right)}\right)$,
3. $t3 = t_0 + \frac{r_c}{2v}\ln\left(-\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{\beta(t_0)}{2}\right)}\right)$;

and either

1. $\frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right) < \alpha(t_1) < -\beta(t_1) \leq 0$, *or*
2. $\exists t_*,\ t_2 < t_* \leq t_1 \mid -\beta(t_*) = \alpha(t_*)$, *and if* $\beta(t_0) \geq \frac{v}{r_c}\frac{\alpha(t_0)}{\frac{v}{r_c}-\kappa}$, *then* $\frac{v}{\kappa r_c}\left(\alpha(t_0) - \beta(t_0)\right) < \alpha(t_*) < 0$ *or*

3. $\exists t_*,\ t_3 < t_* \leq t_1 \mid -\beta(t_*) = \alpha(t_*)$, *and if* $\beta(t_0) < \frac{v}{r_c}\frac{\alpha(t_0)}{\frac{v}{r_c}-\kappa}$, *then*

$$\frac{v}{\kappa r_c}\left(\alpha(t_0)-\beta(t_0)\right) - \left(\beta(t_0) + \frac{v}{\kappa r_c}\left(\alpha(t_0)-\beta(t_0)\right)\right) e^{-\frac{\kappa}{2}\frac{r_c}{v}\ln\left(-\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{\beta(t_0)}{2}\right)}\right)} < \alpha(t_*) < 0$$

or

4. $\exists t_*,\ t_0 \leq t_* \leq t_1 \mid r(t_*) \leq r_c$.

We designate the condition $0 \leq \beta(t) < -\alpha(t) < \frac{\pi}{2}$ with a system state we call C^-.

Lemma 2.9. *If* $\kappa > 2\frac{v}{r_c}$ *and at time* $t = t_0$,

1. $0 < \alpha(t_0) \leq -\beta(t_0) < \frac{\pi}{2}$,
2. $r(t_0) > r_c$;

then

1. $t_1 = t_0 + \frac{r(t_0)}{v}\left(\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{\alpha(t_0)}{4}\right)} - 1\right)$;
2. $t_2 = t_0 + \frac{1}{\kappa}\ln\left(-\frac{\beta(t_0)}{\alpha(t_0)}\right)$;
3. $t_3 = t_0 + \frac{r(t_0)}{v}\ln\left(1 - \frac{\alpha(t_0)}{\sin(\beta(t_0))}\right)$;
4. $r(t) > r(t_0)\ \forall t\ |t_0 < t \leq t_1$.

and either

1. $0 < \alpha(t_1) \leq \frac{\alpha(t_0)}{2}$ *and* $0 < \alpha(t_1) \leq -\beta(t_1) \leq -\beta(t_0) e^{-\left(\kappa\frac{r(t_0)}{v}-2\right)\left(\frac{1+\tan^2\left(\frac{\alpha(t_0)}{4}\right)}{1-\tan^2\left(\frac{\alpha(t_0)}{4}\right)}\right)} < -\beta(t_0)$*; or*
2. $\exists t_*\ |t_2 \leq t_* \leq t_1$ *such that* $0 \leq -\beta(t_*) < \alpha(t_*) < \alpha(t_0)$*; or*
3. $\exists t_*\ |t_3 \leq t_* \leq t_1$ *such that* $\beta(t_0)e^{-\left(\kappa\frac{r(t_0)}{v}-2\right)\ln\left(1-\frac{\alpha(t_0)}{\sin(\beta(t_0))}\right)} \leq \beta(t_*) < \alpha(t_*) = 0$.

We assign the condition $0 < \alpha \leq -\beta \leq \frac{\pi}{2}$ with a system state we call D. Figure 7d shows α, β and their angular velocities while in state D. Figure 5 shows an outline of the proof.

Proof. While in state D, $0 < \alpha \leq -\beta \leq \frac{\pi}{2}$, and therefore

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) \geq 0,$$

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) \leq 0,$$

$$\dot{\alpha} - \dot{\beta} = \kappa\beta < 0,$$

and α shrinks as r grows, and the difference between α and β shrinks as well. Since $-\frac{\pi}{2} \leq \beta \leq -\alpha$ and $0 < \alpha$, we can find an upper bound on r,

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) \leq v\left(1 - 0\right) = v$$

$$\Downarrow$$

$$r(t) \leq r^+(t) = r(t_0) + v\left(t - t_0\right)$$

We can now find an upper bound for α,

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) \leq \frac{v}{r^+(t)}\left(\sin(\beta) - \sin(\alpha)\right) \leq -\frac{v}{r(t_0) + v\left(t - t_0\right)}\sin(\alpha)$$

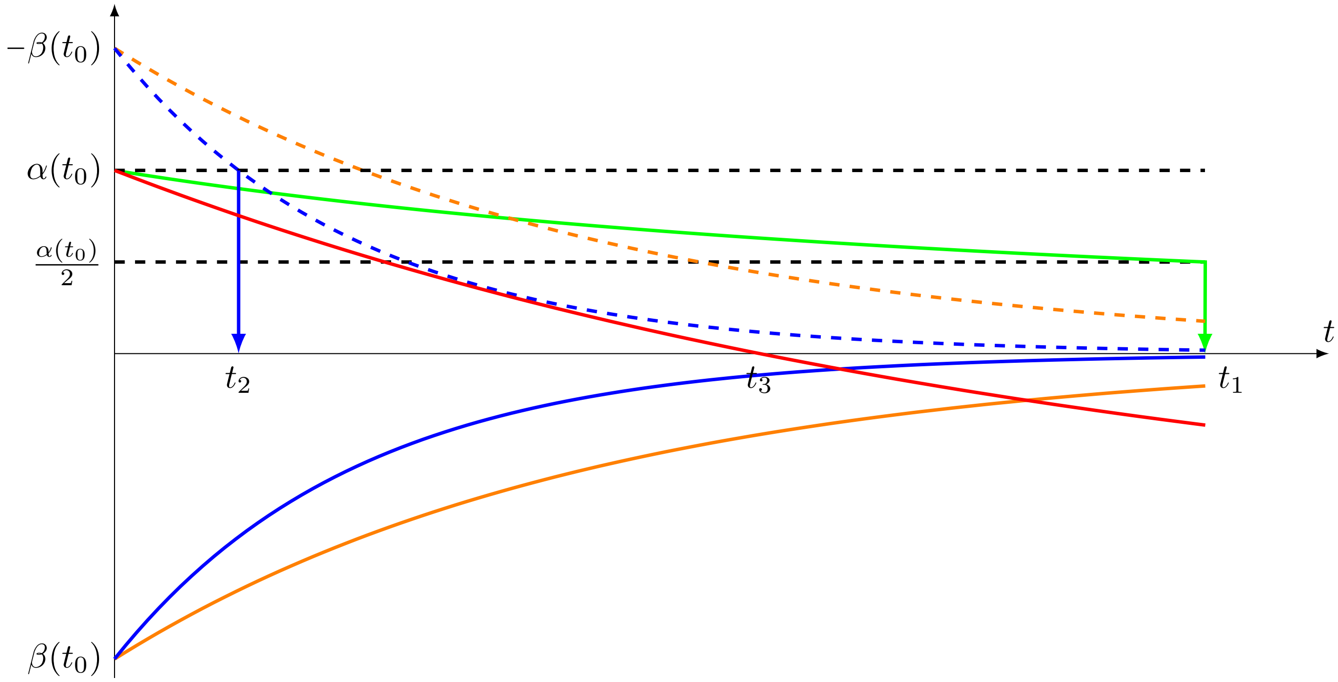


Figure 5: Lemma 2.9 proof outline. After finding an upper bound on r, we use this upper bound to find an upper bound for α and the moment t_1 where the upper bound reaches $\frac{\alpha(t_0)}{2}$. Then, we find the bounds on β, and show that both grow monotonously. Since $|\beta|$ shrinks with time, we can find the lower bound on α, and show that unless leaving the state with $\alpha > -\beta$ or $\alpha = 0$, the angles α, β continue to shrink.

$$\Downarrow$$

$$\frac{d\alpha}{dt} \leq -\frac{v}{r(t_0) + v\,(t - t_0)} \sin(\alpha)$$

$$\Downarrow$$

$$\int \frac{d\alpha}{\sin(\alpha)} \leq -\int \frac{v}{r(t_0) + v\,(t - t_0)} dt.$$

Substituting $w = r(t_0) + v\,(t - t_0)$, $\frac{dw}{dt} = v$, and

$$-\int \frac{v}{r(t_0) + v\,(t - t_0)} dt = -\int \frac{v}{w} \frac{dw}{v}$$

$$\Downarrow$$

$$\ln\left(\tan\left(\frac{\alpha}{2}\right)\right) \leq -\int \frac{dw}{w} = C_0 - \ln\left(r(t_0) + v\,(t - t_0)\right)$$

$$\Downarrow$$

$$\tan\left(\frac{\alpha}{2}\right) \leq e^{(C_0)} \left(e^{(\ln(r(t_0) + v(t - t_0)))}\right)^{-1} = \frac{C_1}{r(t_0) + v\,(t - t_0)}$$

$$\Downarrow$$

$$\alpha(t) \leq \alpha^+(t) = 2 \arctan\left(\frac{r(t_0) \tan\left(\frac{\alpha(t_0)}{2}\right)}{r(t_0) + v\,(t - t_0)}\right).$$

Let t_1 such that $\alpha(t_1) \leq \frac{\alpha(t_0)}{2}$,

$$\frac{\alpha(t_0)}{2} = \alpha^+(t_1) = 2 \arctan\left(\frac{r(t_0) \tan\left(\frac{\alpha(t_0)}{2}\right)}{r(t_0) + v\,(t_1 - t_0)}\right)$$

$$\Downarrow$$

$$t_1 = t_0 + \frac{r(t_0)}{v}\left(\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{\alpha(t_0)}{4}\right)} - 1\right) = t_0 + \frac{r(t_0)}{v}\left(\frac{1+\tan^2\left(\frac{\alpha(t_0)}{4}\right)}{1-\tan^2\left(\frac{\alpha(t_0)}{4}\right)}\right)$$

We can bound $\beta(t)$,

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta \leq -\kappa\beta$$

$$\Downarrow$$

$$\beta(t) \leq \beta^+(t) = \beta(t_0)e^{-\kappa(t-t_0)},$$

and since $r(t_0)$ is the lower bound on r,

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta \geq \frac{v}{r(t_0)}\left(\sin(\beta) - \sin(-\beta)\right) - \kappa\beta \geq \left(\frac{2v}{r(t_0)} - \kappa\right)\beta.$$

$$\Downarrow$$

$$\beta(t) \geq \beta^-(t) = \beta(t_0)e^{\left(\frac{2v}{r(t_0)}-\kappa\right)(t-t_0)}.$$

We can now find t_2, a moment before the first opportunity for $-\beta < \alpha$,

$$\alpha(t_0) = -\beta^+(t_2) = -\beta(t_0)e^{-\kappa(t_2-t_0)}$$

$$\Downarrow$$

$$t_2 = t_0 + \frac{1}{\kappa}\ln\left(-\frac{\beta(t_0)}{\alpha(t_0)}\right).$$

If $\kappa > 2\frac{v}{r_c}$ and $r(t_0) > r_c$, then $r(t_0) > \frac{2v}{\kappa}$, and it is guaranteed that $|\beta(t)|$ shrinks with time; in particular,

$$\beta(t_0)e^{-\left(\kappa\frac{r(t_0)}{v}-2\right)\left(\frac{1+\tan^2\left(\frac{\alpha(t_0)}{4}\right)}{1-\tan^2\left(\frac{\alpha(t_0)}{4}\right)}\right)} \leq \beta(t_1) \leq \beta(t_0)e^{-\kappa\frac{r(t_0)}{v}\left(\frac{1+\tan^2\left(\frac{\alpha(t_0)}{4}\right)}{1-\tan^2\left(\frac{\alpha(t_0)}{4}\right)}\right)},$$

and therefore

$$|\beta(t_0)| > |\beta(t_1)|.$$

Since $\beta(t) > \beta(t_0)\ \forall t > t_0$,

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) \geq \frac{v}{r(t_0)}\left(\sin(\beta(t_0)) - \alpha\right)$$

$$\Downarrow$$

$$\alpha(t) \geq \alpha^-(t) = \sin(\beta(t_0)) + \left(\alpha(t_0) - \sin(\beta(t_0))\right)e^{-\frac{v}{r(t_0)}(t-t_0)},$$

and the first opportunity for $\alpha = 0$ must come after time t_3,

$$0 = \sin(\beta(t_0)) + \left(\alpha(t_0) - \sin(\beta(t_0))\right)e^{-\frac{v}{r(t_0)}(t_3-t_0)}$$

$$\Downarrow$$

$$t_3 = t_0 + \frac{r(t_0)}{v}\ln\left(1 - \frac{\alpha(t_0)}{\sin(\beta(t_0))}\right),$$

at which time

$$\beta > \beta^-(t_3) = \beta(t_0)e^{\left(\frac{2v}{r(t_0)}-\kappa\right)(t_3-t_0)} = \beta(t_0)e^{-\left(\kappa\frac{r(t_0)}{v}-2\right)\ln\left(1-\frac{\alpha(t_0)}{\sin(\beta(t_0))}\right)}.$$

□

Corollary 2.10. *If $\kappa > 2\frac{v}{r_c}$ and at time $t = t_0$,*

1. $0 < -\alpha(t_0) \leq \beta(t_0) < \frac{\pi}{2}$,
2. $r(t_0) > r_c$;

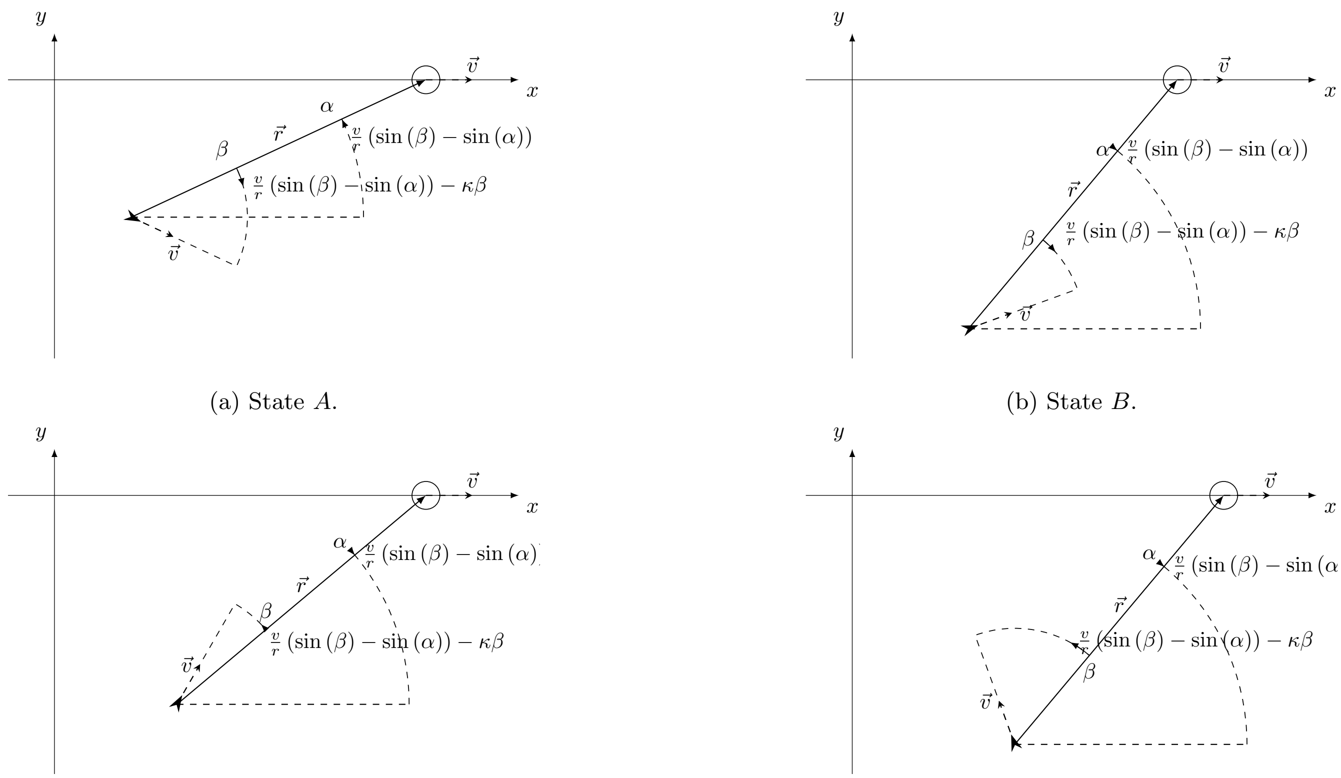
then

1. $t_1 = t_0 + \frac{r(t_0)}{v}\left(\frac{\tan\left(\frac{\alpha(t_0)}{2}\right)}{\tan\left(\frac{\alpha(t_0)}{4}\right)} - 1\right)$;
2. $t_2 = t_0 + \frac{1}{\kappa}\ln\left(-\frac{\beta(t_0)}{\alpha(t_0)}\right)$;
3. $t_3 = t_0 + \frac{r(t_0)}{v}\ln\left(1 - \frac{\alpha(t_0)}{\sin(\beta(t_0))}\right)$;
4. $r(t) > r(t_0)\ \forall t\ |t_0 < t \le t_1$.

and either

1. $0 < -\alpha(t_1) \le -\frac{\alpha(t_0)}{2}$ *and* $0 < -\alpha(t_1) \le \beta(t_1) \le \beta(t_0)e^{-\left(\kappa\frac{r(t_0)}{v}-2\right)\left(\frac{1+\tan^2\left(\frac{\alpha(t_0)}{4}\right)}{1-\tan^2\left(\frac{\alpha(t_0)}{4}\right)}\right)} < \beta(t_0)$; *or*
2. $\exists t_*\ |t_2 \le t_* \le t_1$ *such that* $0 \le \beta(t_*) < -\alpha(t_*) < -\alpha(t_0)$; *or*
3. $\exists t_*\ |t_3 \le t_* \le t_1$ *such that* $0 = \alpha(t_*) < \beta(t_*) \le \beta(t_0)e^{-\left(\kappa\frac{r(t_0)}{v}-2\right)\ln\left(1-\frac{\alpha(t_0)}{\sin(\beta(t_0))}\right)}$.

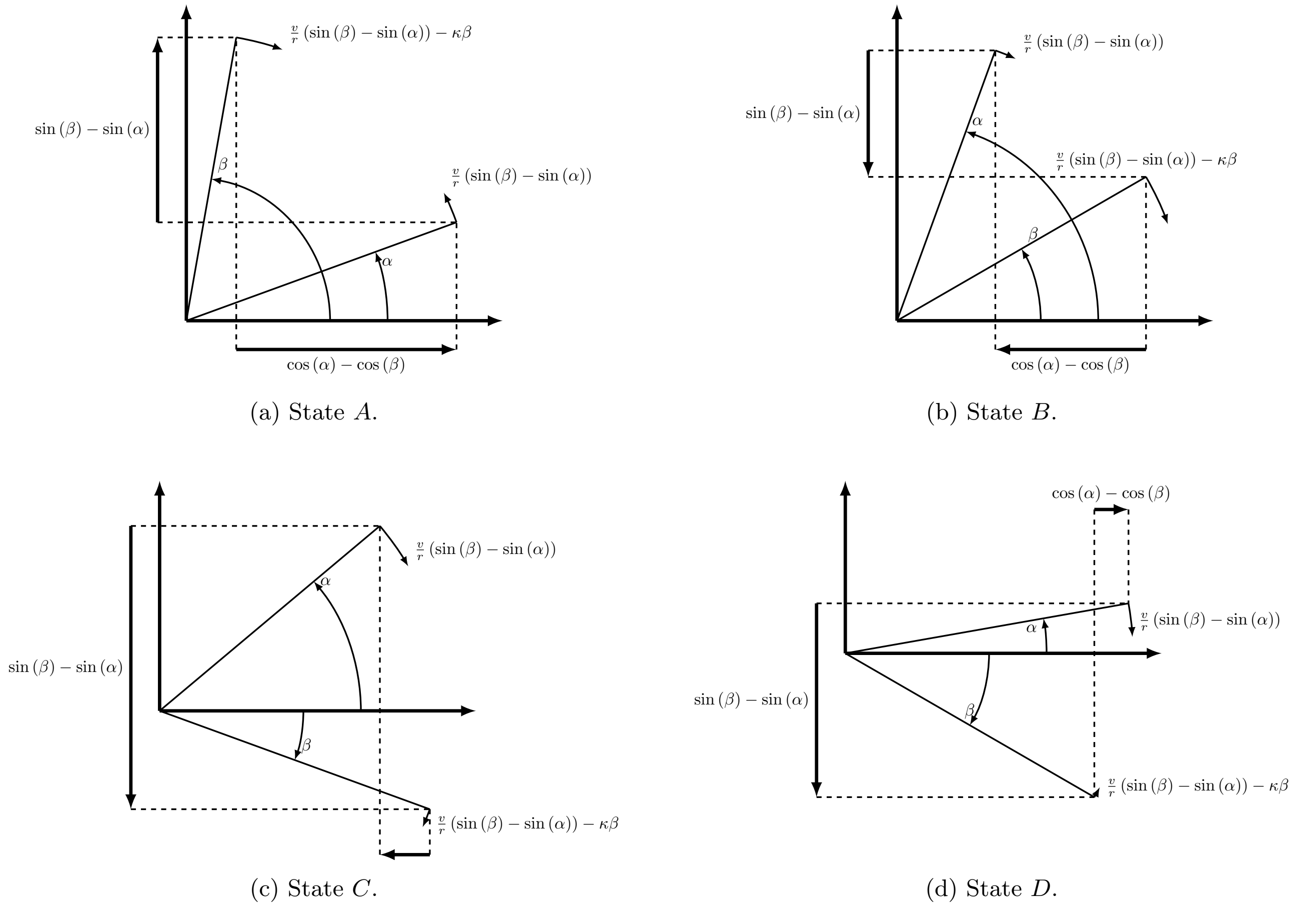
We assign the condition $0 < -\alpha(t) \le \beta(t) < \frac{\pi}{2}$ with a system state we call D^-. Figure 7d shows α, β and their angular velocities while in state D^-.



(a) State A. (b) State B.

(c) State C. (d) State D.

Figure 6: An illustration of a typical configuration on the plain for each system state.



(a) State A.

(b) State B.

(c) State C.

(d) State D.

Figure 7: The primary states, defined by the angle couple α, β, and their velocities.

Theorem 2.11 (Path Following)**.** *If*

1. $\gamma(t) = \max\{|\alpha(t)|, |\beta(t)|\}$,
2. $\kappa > 2\frac{v}{r_c}$, *and*
3. $r(t_0) > r_c$,

then either

1. $\exists T > 0 \mid r(T) \leq r_c$, *or*
2. $\forall \varepsilon \;\big|\; 0 < \varepsilon < \frac{\pi}{2}, \exists T > 0 \mid T < t \Rightarrow \gamma(t) < \varepsilon$.

In other words, a unicycle (Equation 2) in pursuit of a target moving in a straight line (Equation 1) with the bearing-only control law (Equation 3) governing its steering, either captures the target or asymptotically reaches the target's path.

Figure 8 shows the outline of the proof to Theorem 2.11.

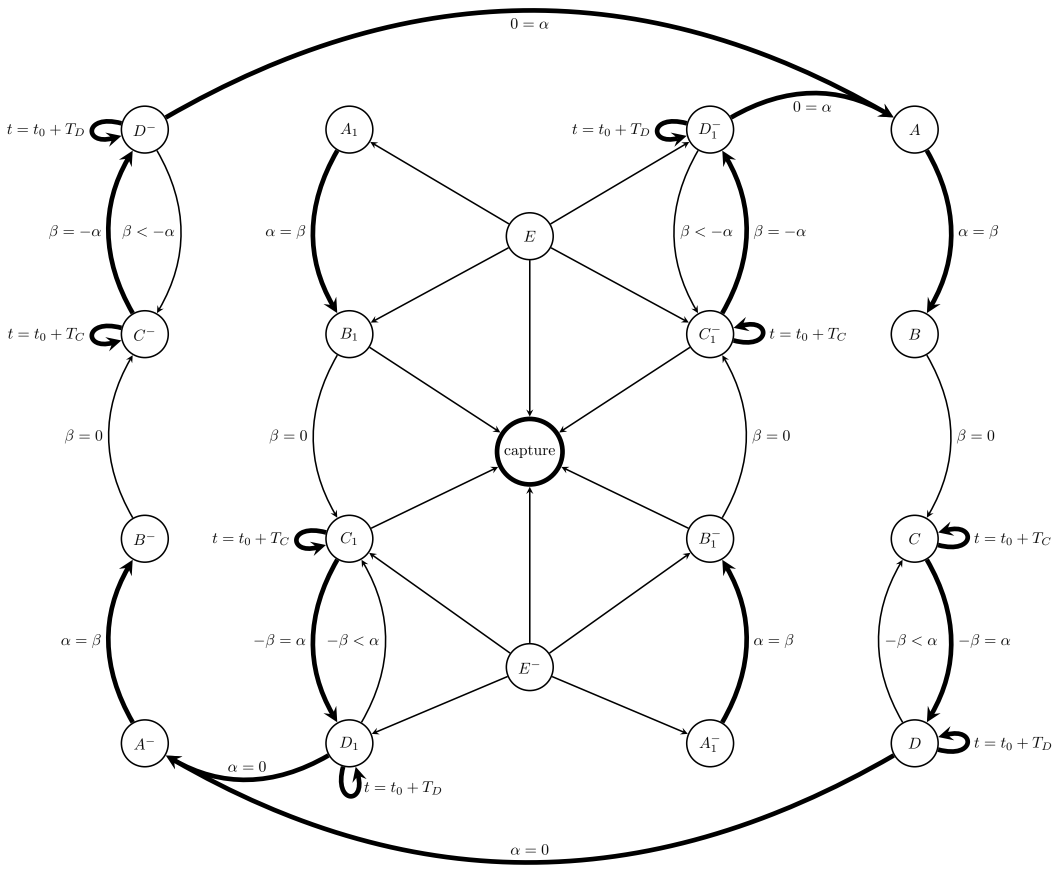


Figure 8: The Pursuit Graph. Each state has a time limit that results in exiting the state when the time limit expires. The bold edges represent transitions that entail a diminishing of γ before exiting the state. Since no loop on the graph is possible without traversing a bold edge, Then γ must shrink every loop. Summing the time upper bounds for each state traversed until the eventual $\gamma < \varepsilon$ results in T.

Proof. Except for the capture state at which the pursuit is concluded, each of the system states has a finite time limit, and the state must transition when the limit lapses, even if re-entering the same state. Any initial condition other than $0 \leq |\alpha| < \frac{\pi}{2}$, $0 \leq |\beta| < \frac{\pi}{2}$ falls into one of the inner states in Figure 8 in finite time, resulting in either capture, remaining in a loop between states C_1, D_1 or their symmetrical states C_1^-, D_1^-, or an eventual exit towards the outer states with the transition into state A or A^- with $\alpha = 0$.

Entering State A with $r = r_0$, $\alpha = 0$ and $\beta = \beta_0$ results in an exit with γ smaller than $\beta^+(t_1)$,

$$\gamma_A \leq \beta^+(t_1) = \beta(t_0)e^{-\left(\kappa - \frac{v}{r(t_0)}\right)(t_1-t_0)} - \frac{\frac{v}{r(t_0)}}{\kappa - \frac{v}{r(t_0)}} \sin\left(\alpha(t_0)\right)\left(1 - e^{-\left(\kappa - \frac{v}{r(t_0)}\right)(t_1-t_0)}\right)$$

$$= \beta(t_0)e^{-\left(\kappa - \frac{v}{r(t_0)}\right)(t_1-t_0)} = \beta(t_0)e^{-\left(\kappa - \frac{v}{r(t_0)}\right)\left(t_0 + \frac{1}{\kappa}\ln\left(\frac{2}{1+\frac{\alpha(t_0)}{\beta(t_0)}}\right) - t_0\right)} = \beta(t_0)\frac{1}{2^{\frac{1}{\kappa}\left(\kappa - \frac{v}{r(t_0)}\right)}}$$

$$\Downarrow$$

$$\gamma_A \leq 2^{\frac{v}{\kappa r(t_0)}} \frac{\beta(t_0)}{2}. \tag{9}$$

and the state transitions to B with $\alpha = \beta < \gamma_A$.

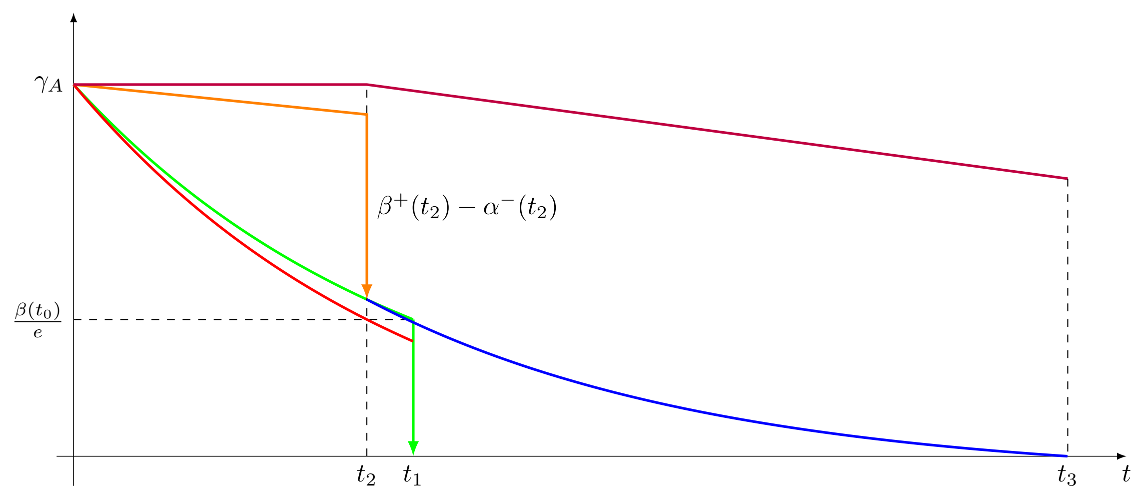


Figure 9: State B. Similar to Figure 3, the initial condition $\alpha_0 = \beta_0 = \gamma_A$ from the transition from state A results in exiting state B with $\gamma_B < \gamma_A$.

Exiting state B, γ is less than γ_A, and $r < r^+(t_2)$, see Figure 9.

$$r^+(t_2) = r_A + v\left(\cos\left(\alpha^-(t_2)\right) - \cos\left(\beta^+(t_2)\right)\right)(t_2 - t_0)$$

and the state transitions to state C with $\alpha = \gamma_B < \gamma_A$ and $\beta = 0$. Figure 10 shows the evolution of α and β after entering state C from state B.

Any consecutive loop between state C and itself follows the schema presented in Figure 4. Given $\kappa > 2\frac{v}{r_c}$, each iteration shrinks γ, the greater the κ the greater the step in each iteration,

$$\gamma_C^{i+1} < 2\frac{v}{\kappa r_c}\gamma_C^i < \gamma_C^i.$$

Ultimately, $\gamma_C < \frac{v}{\kappa r_c}\gamma_B < \frac{v}{\kappa r_c}\gamma_A$ for any amount of self-loops, and since $\dot{\alpha} < \dot{\beta}$ while in state C, a transition to state D must occur in finite time, with $\alpha = -\beta = \gamma_C < \frac{v}{\kappa r_c}\gamma_A$.

As shown in Figure 11, a transition back to state C is possible immediately, with $\gamma_D^C = \gamma_C$. Yet on return from state C, $\gamma < 2\frac{v}{\kappa r_c}\gamma_C$, as discussed above, and the result of the $D \to C \to D$ loop is $\gamma_C^{i+1} < 2\frac{v}{\kappa r_c}\gamma_D^C = 2\frac{v}{\kappa r_c}\gamma_C^i$ for

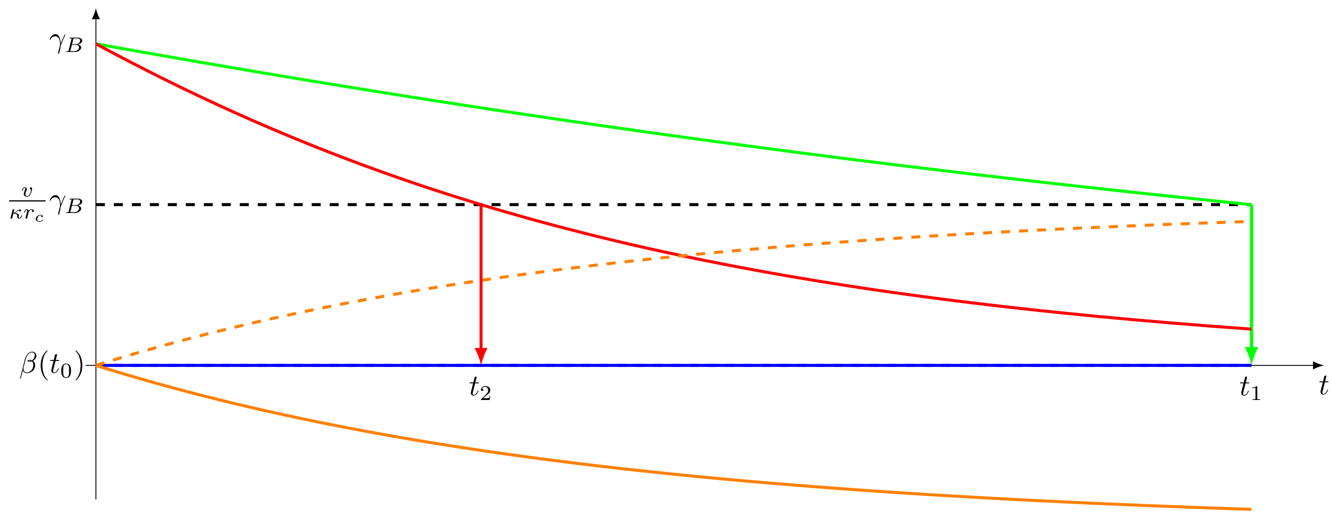


Figure 10: State C after state B. Similar to Figure 4, with the initial conditions $\alpha_0 = \gamma_B$ and $\beta_0 = 0$. By t_1, a transition must have happened, either to state D or re-entry to state C, with $\gamma_C \leq \frac{v}{\kappa r_c}\gamma_B$.

each return to state D from C. If not returning to state C, the first opportunity to transition out of D is to state A^-, which happens at t_3, when

$$\gamma_D^A < -\beta^-(t_3) = -\beta(t_0)e^{-\left(\kappa\frac{r(t_0)}{v}-2\right)\ln\left(1-\frac{\alpha(t_0)}{\sin(\beta(t_0))}\right)} = \gamma_C\left(\frac{-\sin(\gamma_C)}{-\sin(\gamma_C)-\gamma_C}\right)^{\left(\kappa\frac{r(t_0)}{v}-2\right)}$$

$$= \gamma_C\left(\frac{\sin(\gamma_C)}{\sin(\gamma_C)+\gamma_C}\right)^{\left(\kappa\frac{r(t_0)}{v}-2\right)} < \gamma_C$$

$$\Downarrow$$

$$\gamma_D^A < 2^{\left(2-\kappa\frac{r(t_0)}{v}\right)}\gamma_C = \frac{4}{2^{\kappa\frac{r(t_0)}{v}}}\gamma_C \tag{10}$$

Any other transitions, including self-loops or other transitions back and forth, shrink γ more, to a value less than γ_D^A.
The loop $A \to B \to C \to D \to A^- \to B^- \to C^- \to D^- \to A$, as seen in Figure 8, results in

$$\gamma_A^{i+1} \leq 2^{\frac{v}{\kappa r_c}}\frac{\gamma_D^A}{2} \leq \frac{2^{\frac{v}{\kappa r_c}}}{2}\gamma_C\left(\frac{\sin(\gamma_C)}{\sin(\gamma_C)+\gamma_C}\right)^{\left(\kappa\frac{r_c}{v}-2\right)} \leq \frac{2^{\frac{v}{\kappa r_c}}}{2}\gamma_C \leq \frac{2^{\frac{v}{\kappa r_c}}}{2}\frac{v}{\kappa r_c}\gamma_A^-$$

$$\leq \frac{2^{\frac{2v}{\kappa r_c}}}{4}\left(\frac{v}{\kappa r_c}\right)^2\gamma_A^i < \frac{\gamma_A^i}{8}$$

$$\Downarrow$$

$$\gamma_A^{i+1} < \frac{\gamma_A^i}{8} < \frac{\gamma_A^{i-1}}{8^2} < \frac{\gamma_A^{i-2}}{8^3} < \cdots < \frac{\gamma_A^0}{8^{(i+1)}}$$

$$\Downarrow$$

$$\lim_{i\to\infty}\gamma_A^{i+1} < \lim_{i\to\infty}\frac{\gamma_A^0}{8^{(i+1)}} = 0.$$

In conclusion, given a system comprising a unicycle agent (Equation 2) in pursuit of a target moving in a straight line (Equation 1) with the bearing-only control law (Equation 3) governing the agent's steering, the system's evolution through time can be mapped to the graph shown in Figure 8. Since all states other than the capture state are timed, the system must perpetually transition the states if the agent does not capture the target before $\alpha = 0$ for the first time. This perpetual transitioning entails transitioning in loop paths in the graph, and we have shown

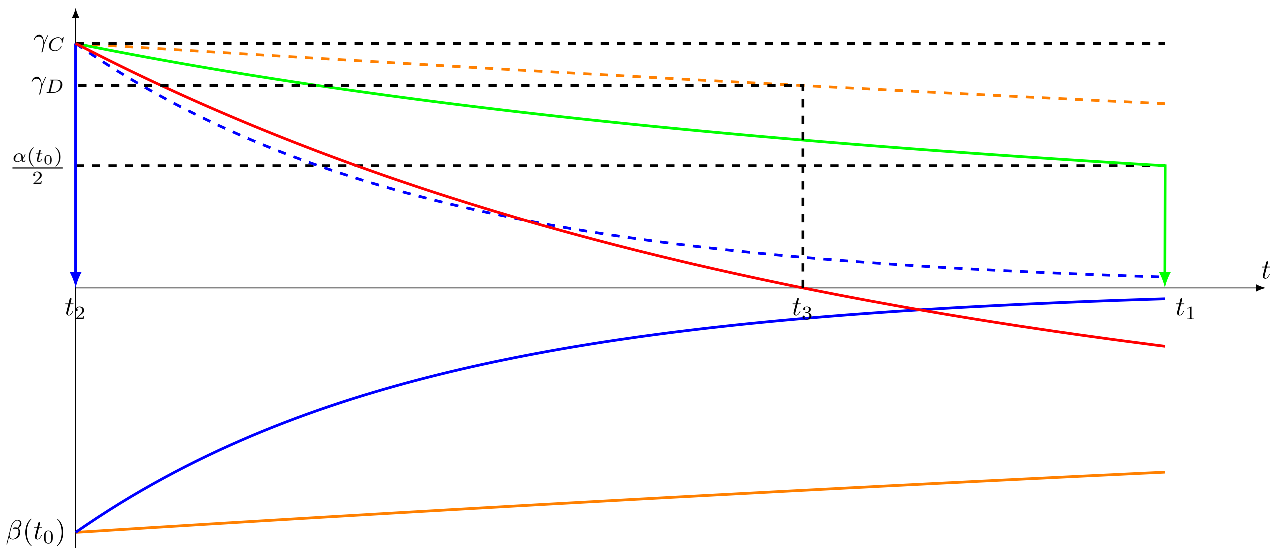


Figure 11: State D after state C. Similar to Figure 5, with the initial conditions $\alpha_0 = -\beta_0 = \gamma_C$. By t_1, a transition must have happened, either to state A, back to C, or re-entry to state D.

that any of these paths must include at least one type of loop that decreases in γ, $\forall \kappa > 2\frac{v}{r_c}$. Therefore, either $\exists T > 0 \,|r(T) \leq r_c$, or $\forall \varepsilon > 0, \exists T > 0 \,|\gamma(t) < \varepsilon \; \forall t > T$, and the agent either captures the target or asymptotically reaches the target's path. ☐

Theorem 2.12 (Finite Distance of Path Following)**.** *If*

1. $\gamma(t) = \max\{|\alpha(t)|, |\beta(t)|\}$,
2. $\kappa > 2\frac{v}{r_c}$, *and*
3. $r(t_0) > r_c$,

then $\exists R,\ 0 < R < \infty \mid r(t) < R\ ,\ \forall t$.

In other words, the distance between a target moving in a straight line (Equation 1) and a unicycle agent (Equation 2) in pursuit of the target with the bearing-only control law (Equation 3) governing its steering, is bound from above by a finite value.

Proof. From the previous discussion we can assert that the only states where the distance between the agent and the target grows are states A, D and their variations A^-, A_1, D_1^- etc. While the transition from state A is exclusive to state B, state D can transition towards state A^-, state C, and self-loop to state D.

Leaving state D increases the distance between the pursuing agents and the target by the maximum relative velocity times the maximal time spent in state D. From similar considerations to those in the proof of Theorem 2.11, the distance on exit of state D, r_0^D, as function of the distance upon entry from state C, r_C and the entry γ_C, is

$$r_0^D < r_C + v\left(\cos(0) - \cos(\gamma_C)\right)(t_1 - t_0)$$

$$= r_C + 2v\sin^2\left(\frac{\gamma_C}{2}\right)\frac{r_C}{v}\left(\frac{\tan\left(\frac{\gamma_C}{2}\right)}{\tan\left(\frac{\gamma_C}{4}\right)} - 1\right) = r_C\left(1 + 2\sin^2\left(\frac{\gamma_C}{2}\right)\left(\frac{\sin\left(\frac{\gamma_C}{2}\right)}{\cos\left(\frac{\gamma_C}{2}\right)}\frac{\cos\left(\frac{\gamma_C}{4}\right)}{\sin\left(\frac{\gamma_C}{4}\right)} - 1\right)\right)$$

$$= r_C\left(1 + 2\sin^2\left(\frac{\gamma_C}{2}\right)\left(\frac{2\sin\left(\frac{\gamma_C}{4}\right)\cos\left(\frac{\gamma_C}{4}\right)}{2\cos^2\left(\frac{\gamma_C}{4}\right) - 1}\frac{\cos\left(\frac{\gamma_C}{4}\right)}{\sin\left(\frac{\gamma_C}{4}\right)} - 1\right)\right)$$

$$= r_C\left(1 + 2\sin^2\left(\frac{\gamma_C}{2}\right)\left(\frac{2\cos^2\left(\frac{\gamma_C}{4}\right) - 2\cos^2\left(\frac{\gamma_C}{4}\right) + 1}{2\cos^2\left(\frac{\gamma_C}{4}\right) - 1}\right)\right)$$

$$= r_C\left(1+2\left(\frac{\sin^2\left(\frac{\gamma_C}{2}\right)}{\cos\left(\frac{\gamma_C}{2}\right)}\right)\right) = r_C\left(1+2\sin\left(\frac{\gamma_C}{2}\right)\tan\left(\frac{\gamma_C}{2}\right)\right)$$

$$\Downarrow$$

$$r_0^D < r_C\left(1+2\sin\left(\frac{\gamma_C}{2}\right)\tan\left(\frac{\gamma_C}{2}\right)\right). \tag{11}$$

In case of a self-loop, $\alpha_1 < \frac{\gamma_C}{2}$, and from Equation 10,

$$\gamma_1^D < 2^{\left(2-\kappa\frac{r_0^D}{v}\right)}\gamma_C$$

$$\Downarrow$$

$$r_1^D < r_0^D + 2v\sin^2\left(\frac{\gamma_1^D}{2}\right)\frac{r_C}{v}\left(\frac{\tan\left(\frac{\alpha_1}{2}\right)}{\tan\left(\frac{\alpha_1}{4}\right)}-1\right) = r_0^D\left(1+2\sin^2\left(\frac{\gamma_1^D}{2}\right)\left(\frac{\tan\left(\frac{\alpha_1}{2}\right)}{\tan\left(\frac{\alpha_1}{4}\right)}-1\right)\right)$$
$$< r_0^D\left(1+2\sin^2\left(2^{\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{4}\right)}{\tan\left(\frac{\gamma_C}{8}\right)}-1\right)\right);$$

similarly for the second self-loop iteration,

$$\gamma_2^D < 2^{\left(2-\kappa\frac{r_1^D}{v}\right)}\gamma_1^D < 2^{\left(2-\kappa\frac{r_1^D}{v}\right)\left(2-\kappa\frac{r_0^D}{v}\right)}\gamma_C$$

$$\Downarrow$$

$$r_2^D < r_1^D\left(1+2\sin^2\left(2^{\left(2-\kappa\frac{r_1^D}{v}\right)\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{8}\right)}{\tan\left(\frac{\gamma_C}{16}\right)}-1\right)\right)$$
$$< r_0^D\left(1+2\sin^2\left(2^{\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{4}\right)}{\tan\left(\frac{\gamma_C}{8}\right)}-1\right)\right)\left(1+2\sin^2\left(2^{\left(2-\kappa\frac{r_1^D}{v}\right)\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{8}\right)}{\tan\left(\frac{\gamma_C}{16}\right)}-1\right)\right);$$

and the general iteration,

$$r_N^D < r_0^D\prod_{n=1}^{N}\left(1+2\sin^2\left(2^{\Pi_{m=0}^{n-1}\left(2-\kappa\frac{r_m^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+1)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}-1\right)\right)$$
$$< r_0^D\prod_{n=1}^{N}\left(1+2\sin^2\left(2^{n\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+1)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}-1\right)\right) = R_N.$$

Consider the difference in distance between two consecutive loops,

$$R_N - R_{N-1} = R_{N-1}\left(\left(1+2\sin^2\left(2^{n\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+1)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}-1\right)\right)-1\right)$$
$$= R_{N-1}\left(2\sin^2\left(2^{n\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+1)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}-1\right)\right) = D_N.$$

The maximal distance gained during the state D self loop, if such a distance L_D exists, is therefore less than the sum of differences

$$L_D < R_\infty - R_1 < \sum_{n=1}^{\infty} D_n.$$

Applying d'Alembert's ratio test on the series produces the following result,

$$\lim_{n\to\infty}\left|\frac{D_{n+1}}{D_n}\right| = \lim_{n\to\infty}\left|\frac{R_N\left(2\sin^2\left(2^{(n+1)\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+3)}}\right)}-1\right)\right)}{R_{N-1}\left(2\sin^2\left(2^{n\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+1)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}-1\right)\right)}\right|$$

$$= \lim_{n\to\infty} \frac{R_{N-1}\left(1+2\sin^2\left(2^{n\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+1)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}-1\right)\right)\left(2\sin^2\left(2^{(n+1)\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+3)}}\right)}-1\right)\right)}{R_{N-1}\left(2\sin^2\left(2^{n\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+1)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}-1\right)\right)}$$

$$= \lim_{n\to\infty}\left(\frac{2\sin^2\left(2^{(n+1)\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+3)}}\right)}-1\right)}{2\sin^2\left(2^{n\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+1)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}-1\right)} + 2\sin^2\left(2^{(n+1)\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+3)}}\right)}-1\right)\right)$$

$$= \lim_{n\to\infty}\frac{2\sin^2\left(2^{(n+1)\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+3)}}\right)}-1\right)}{2\sin^2\left(2^{n\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)\left(\frac{\tan\left(\frac{\gamma_C}{2^{(n+1)}}\right)}{\tan\left(\frac{\gamma_C}{2^{(n+2)}}\right)}-1\right)} = \lim_{n\to\infty}\frac{2\left(2^{(n+1)\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)^2}{2\left(2^{n\left(2-\kappa\frac{r_0^D}{v}\right)}\frac{\gamma_C}{2}\right)^2}$$

$$= \lim_{n\to\infty}\frac{2^{2(n+1)\left(2-\kappa\frac{r_0^D}{v}\right)}}{2^{2n\left(2-\kappa\frac{r_0^D}{v}\right)}} = \lim_{n\to\infty}\frac{2^{2n\left(2-\kappa\frac{r_0^D}{v}\right)}2^{2\left(2-\kappa\frac{r_0^D}{v}\right)}}{2^{2n\left(2-\kappa\frac{r_0^D}{v}\right)}} = 2^{2\left(2-\kappa\frac{r_0^D}{v}\right)} < 2^{4\left(1-\frac{r_0^D}{r_c}\right)} < \frac{1}{16} < 1,$$

therefore a finite L_D exists. In the case of loop between states C and D, seeing that entering and exiting state C shrinks both r and α, and as consequence shrinks γ as well, we can consider the $D \to C \to D$ loop as an intermission in the D self loop after which the loop restarts with smaller r_C and γ_C than the last r_n^D and γ_n^D, therefore any conclusion of such a loop would result in less relative distance gained compared to an alternative D self loop that resulted in the same decrease in γ or during the same period of time. We conclude the state D discussion by stating that the maximal gain in r for the $D \to C \to D$ and the D self loop must therefore be less than L_D.

We shall now assess the contribution of state A to the growth of r. Let us assume that all other states in the state graph (Figure 8) make no contribution to γ and r, and if any changes occur, they happen exclusively in state A. In this case there is no actual meaning to leaving the state, other than having α miraculously return to 0 for the next iteration. If at the initial entry, $r = r_0$, $\beta = \beta_0$, and $\alpha = 0$, then

$$\gamma(t) < \beta^+(t) = \beta_0 e^{-\left(\kappa-\frac{v}{r_0}\right)(t-t_0)} - \frac{\frac{v}{r_0}}{\kappa-\frac{v}{r_0}}\sin(0)\left(1-e^{-\left(\kappa-\frac{v}{r_0}\right)(t-t_0)}\right)$$

$$= \beta_0 e^{-\left(\kappa-\frac{v}{r_0}\right)(t-t_0)} < \beta_0 e^{-\left(\kappa-\frac{v}{r_c}\right)(t-t_0)} = \gamma_A^+(t).$$

Since the $\gamma_A^+(t)$ rate of decay is constant, and remains the same regardless of the initial conditions on entering the state other than $\beta_0 = \gamma_A^+(t_0)$, we can arbitrarily choose when to leave and re-enter the state, so we choose $t_1 = t_0 + \frac{1}{\kappa}\ln(2)$, and the change per iteration for γ becomes

$$\gamma_n^A = 2^{\left(\frac{v}{\kappa r_c}-1\right)}\gamma_{n-1}^A = 2^{n\left(\frac{v}{\kappa r_c}-1\right)}\beta_0. \tag{12}$$

The upper bound on the distance between pursuing agent and target on the nth iteration becomes

$$r_n^A = r_{n-1}^A + \frac{v}{\kappa}\left(1-\cos\left(\gamma_{n-1}^A\right)\right)\ln(2),$$

and the distance gained per iteration is

$$r_n^A - r_{n-1}^A = \frac{v}{\kappa}\left(1-\cos\left(\gamma_{n-1}^A\right)\right)\ln(2) = 2\ln(2)\frac{v}{\kappa}\sin^2\left(\frac{\gamma_{n-1}^A}{2}\right)$$

$$< 2\ln(2)\frac{vr_c}{2v}\left(\frac{\gamma_{n-1}^A}{2}\right)^2 = \frac{r_c}{4}\ln(2)\left(2^{(n-1)\left(\frac{v}{\kappa r_c}-1\right)}\beta_0\right)^2 = \frac{r_c}{4}\ln(2)\,2^{2(n-1)\left(\frac{v}{\kappa r_c}-1\right)}\beta_0^2$$

$$< \frac{r_c}{4}\ln(2)\,2^{2(n-1)\left(\frac{1}{2}-1\right)}\beta_0^2 = \frac{r_c}{4}\ln(2)\,\beta_0^2 2^{(1-n)} = \frac{\ln(2)}{2}r_c\beta_0^2\frac{1}{2^n} = A_n.$$

Summing all A_n will result in an upper bound to all the contribution to the growth of r by all variances of state A, on any possible path on the state graph $\mathcal{G}$ (Figure 8). Let us denote this bound by L_A,

$$L_A = \sum_{n=1}^{\infty} A_n = r_{\infty}^{A} - r_1^{A}.$$

$$\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \left(\frac{\ln(2)}{2} r_c \beta_0^2 \frac{1}{2^n} \right) = \frac{\ln(2)}{2} r_c \beta_0^2 \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

Notice that

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} - \frac{1}{2}$$
$$\Downarrow$$
$$\frac{1}{2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^n}$$
$$\Downarrow$$
$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

and therefore

$$L_A = \frac{\ln(2)}{2} r_c \beta_0^2. \tag{13}$$

Given the previous analysis of the state graph $\mathcal{G}$ (Figure 8), the only states that add to the initial distance $r(t=0)$ are states A, D, and their variants. The maximal contribution of any of those states combined is less than $R = L_A + L_D$, therefore,

$$\exists R,\ 0 < R < \infty \mid r(t) < R\ ,\ \forall t.$$

□

3 Conclusion

In this technical report we investigated the unicycle pursuit problem and mapped its system states to a state graph $\mathcal{G}$. By analyzing the graph's possible paths, we found that regardless of initial conditions, if the controller's gain (Equation 3) is above a certain value $\left(\kappa > \frac{2v}{r_c}\right)$, then a pursuing agent must either capture or track the target, and that the distance between agent and target has a finite upper bound.